The Leontief Open Model

1. An economy has two industries: energy and material.
   To produce \$1 of energy requires 90¢ of energy and 30¢ of material.
   To produce \$1 of material requires 20¢ of energy and 30¢ of material.
   (a) Given an external demand for \$2000 of energy and \$1000 of material, how much of each industry should be produced to meet it?
   (b) Is the economy productive? Justify your answer.
   (c) Find the internal consumption when demand is met.

2. An economy has two industries: iron and steel. To produce \$1 of iron requires 20¢ of iron and 10¢ of steel. To produce \$1 of steel requires 70¢ of iron and 40¢ of steel.
   (a) Given an external demand for \$8200 of Iron and \$4100 of Steel, how much of each industry should be produced to meet it?
   (b) Which industries are profitable?

3. An economy has two industries: goods and services. To produce \$1 of goods requires 60¢ of goods and 50¢ of services. To produce \$1 of services requires 30¢ of goods and 10¢ of services.
   (a) If there is an external demand for \$6300 of goods and \$8400 of services, how much of each industry should be produced to meet it?
   (b) Is the economy productive? Justify your answer.

4. An economy has two industries: services and manufacturing. To produce \$1 of services requires 20¢ of services and 40¢ of manufacturing. To produce \$1 of manufacturing requires 30¢ of services and 10¢ of manufacturing.
   (a) If there is an external demand for \$900 of services and \$1500 of manufacturing, how much of each industry should be produced to meet it?
   (b) Find the internal consumption when demand is met.
   (c) Which industries, if any, are profitable.

5. For each of the consumption matrices below, determine which industries are profitable and whether the economy is productive.
   (a) $C = \begin{bmatrix} .8 & .3 \\ .1 & .6 \end{bmatrix}$ (b) $C = \begin{bmatrix} .8 & .1 \\ .3 & .6 \end{bmatrix}$ (c) $C = \begin{bmatrix} .8 & .1 \\ .9 & .6 \end{bmatrix}$

6. Suppose that an economy consists of three industries: a computing service, a statistical service, and an engineering service. For each \$1 of computing that is provided, 30¢ is spent on computing, 10¢ on statistical services and 30¢ on engineering. For each \$1 on statistical service, 20¢ is spent on computing, 40¢ on statistics, and 20¢ on engineering. Each \$1 in engineering takes 30¢ in computing, 10¢ in statistical services, and 30¢ in engineering. Suppose there is an external demand for \$1000 in computing, \$1500 in statistical services, and \$1800 in engineering.
   (a) Compute $\det(I - C)$.
   (b) Compute $(I - C)^{-1}$.
   (c) How much should each industry produce to meet the demand?

Graphical Linear Programming (The Graphical Method For Optimization)

7. Solve the linear program by graphing the feasibility region.
   Minimize $z = 7x - 14y$
   subject to $3x + 4y \leq 12$
   $x - y \geq 0$
   $0 \leq x, \quad 0 \leq y$

8. Solve the linear program by graphing the feasibility region.
   Maximize $z = -x + y$
   subject to $x + y \geq 9$
   $x - 4y \geq 4$
   $2x + y \leq 26$
   $x \geq 0, \quad y \geq 0$

9. Solve the linear program by graphing the feasibility region.
   Maximize $z = 7x - 4y$
   subject to $7x - 3y \geq 4$
   $7x + y \leq 57$
   $y \geq 1$
   $y \leq 8$

10. Solve the linear program by graphing the feasibility region.
    Minimize $z = 3x - 2y$
    subject to $2x + y \geq 4$
    $x - y \leq 4$
    $-3x + 7y \leq 0$
    $x \geq 0, \quad y \geq 0$

11. Solve the linear program by graphing the feasibility region.

$$\begin{array}{lrcrcrcl} \textbf{Minimize} & z & = & x & - & y & & \\ \text{subject to} & & & x & + & 2y & \geq & 16 \\ & & & 2x & - & y & \geq & 12 \\ & & & & & y & \leq & 12 \\ & & & x & \geq 0, & y & \geq 0 & \end{array}$$

12. Solve the linear program by graphing the feasibility region.

$$\begin{array}{lrcrcrcl} \textbf{Minimize} & z & = & -2x & + & 3y & & \\ \text{subject to} & & & x & + & 4y & \geq & 8 \\ & & & 6x & - & 9y & \leq & 12 \\ & & & & & x & \leq & 6 \\ & & & x & \geq 0, & y & \geq 0 & \end{array}$$

13. You are selling martinis at an event to raise funds for charity. A dry martini is $5/3$ oz. of Gin and $1/3$ oz. of Vermouth. A medium martini is $3/2$ oz. Gin and $1/2$ oz. Vermouth. You have available 60 oz. of Gin and 16 oz. of Vermouth. You can sell dry martinis for \$8 each and medium martinis for \$10 each. How many of each kind of martini should be sold to maximize revenue? Name your variables, set up a linear program, and solve by sketching the feasibility region.

14. A retailer offers two home-made coffee blends that contain three kinds of coffee beans: Brazilian, Colombian and Ethiopian, sold in 500 g bags. A bag of the Freaking Strong blend contains 100 g Brazilian, 300 g Colombian and 100 g Ethiopian beans. A bag of the Relax Mild blend contains 100 g Brazilian, 100 g Colombian and 300 g Ethiopian beans. Moreover, the retailer gets a rebate if he orders simultaneously at least 6 kg Brazilian, 10 kg Colombian and 10 kg Ethiopian coffee beans on a weekly basis. All 500 g coffee bags are sold at the same price, however the Ethiopian coffee beans cost more than the other two. Therefore the production cost of the two blends is different: \$3 for the Freaking Strong blend and \$4 for the Relax Mild blend. The retailer wants to offer both coffee blends while minimizing the production cost.
    (a) Name variables and set up a linear program that represents this situation.
    (b) How many bags of each coffee blend should be prepared each week to minimize the production cost?
    (c) What quantity of each kind of coffee beans must be bought each week to prepare the number of bags determined in part (b)?

15. An elementary school principal is organizing an outing for 180 students. A local bus company offers 21-seat buses for \$300 and 33-seat buses for \$400. The students must be accompanied by one adult for 21-seat buses and by three adults for 33-seat buses. The principal can count on a maximum of twelve adults to accompany the students and she wishes to minimize the transportation cost for the outing. Name variables, set up a linear program and determine the number of each kind of bus the principal must rent as well as the associated cost.

16. The Furniture Factory plans on manufacturing two new kitchen shelves to occupy down time in its workshops. Production of the first model requires 2 time intervals in the cutting workshop, 3 time intervals in the assembly workshop and 3 time intervals in the sanding workshop. Production of the second model requires 3 time intervals in the cutting workshop, 2 time intervals in the assembly workshop and 1 time interval in the sanding workshop. The monthly down time report reveals that there are 240 time intervals available in the cutting workshop, 210 in the assembly workshop and 180 in the sanding workshop. Moreover there are no limitations on building materials since the company can easily procure that it needs.
    (a) Given that the profit per kitchen shelf is \$80 for the first model and \$60 for the second model, determine the number of each model that can be made to maximize the monthly profit.
    (b) If the profit per kitchen shelf was \$90 for the first model and \$60 for the second model, what would be the number of each model that could be made?

17. A popular cleaning brand produces two housecleaners, EnviroShine and Nature's Scent. Both cleaner contain the same basic ingredients but in different proportions. To protect their trade secret, we will refer to the ingredients as ingredient I_1, I_2 and I_3. These cleaners are sold in 1 L containers. The production of 1 L of EnviroShine requires 0.4 L of I_1, 0.3 L of I_2 and 0.3 L of I_3; the production of 1 L of Nature's Scent requires 0.5 L of I_1, 0.2 L of I_2 and 0.3 L of I_3. The company's suppliers can guarantee a weekly supply of 94 L of I_1, 51 L of I_2 and 60 L of I_3. The profit margin for every EnviroShine bottle is \$1.50 and \$1.20 for every Nature's Scent bottle.
    (a) How many litres of each housecleaner does the company need to produce weekly to maximize its profit?
    (b) What quantities of the 3 ingredients should the company purchase weekly to avoid needing to store the surplus?
    (c) The distribution service informs the director of production that they are unable to sell all the EnviroShine's stock. The reports for the last months show that the distributers only sold 70 L of this product per week. Given this information, establish a new weekly production.
    (d) Does the revised work plan require that the weekly purchasing be modified? What quantity of each ingredients must the company now purchase weekly?

18. In each of the following cases, find the state vector $\mathbf{x_3}$, given the transition matrix and state vector provided.

    (a) $P = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$ and $\mathbf{x_0} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

    (b) $P = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$ and $\mathbf{x_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    (c) $P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.6 & 0.8 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$

19. For each of the following regular transition matrices, find the associated steady state vector.

    (a) $\begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$

    (b) $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

    (c) $\begin{bmatrix} 0.3 & 0.1 & 0.5 \\ 0.4 & 0.8 & 0.1 \\ 0.3 & 0.1 & 0.4 \end{bmatrix}$

20. (a) Given $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$, find P^3.

    (b) Use your result from part (a) to answer the following question:

    If my cat is awake at some moment, there is a $1/3$ chance that she will be awake an hour from then. If my cat is asleep at some moment, there is a $1/2$ chance that she will be awake an hour from then.

    My cat is awake right now. What is the chance that she will be asleep three hours from now?

21. At a certain company, on any given day some employees are at work and the rest are absent. It is known that if an employee is at work today, there is a 80% chance that they will be at work tomorrow (and a 20% chance that they will be absent tomorrow), and if the employee is absent today, there is a 75% chance that they will be at work tomorrow (and a 25% chance that they will be absent tomorrow).

    In the long term, what % of employees do we expect to be at work each day, and what % of employees do we expect to be absent?

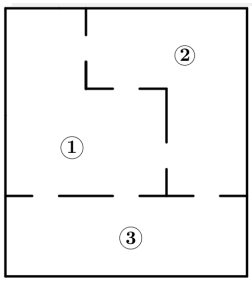
22. In a certain urban area the overall population remains stable, but every year 10% of the people who live in the urban core move out to the suburbs, and 25% of the people who live in the suburbs move in to the urban core.

    In the long term, what % of people do we expect to live in the urban core, and what % do we expect to live in the suburbs?

23. At a certain college, on any given day some students buy lunch in the cafeteria and the rest bring a lunch from home. It is known that if a student eats in the cafeteria today, there is a 34% chance that they will eat in the cafeteria tomorrow, and if they bring a lunch from home today, there is a 44% chance that they will bring a lunch from home tomorrow.

    On a day in the distant future, what % of students do we expect to be buying their lunch in the cafeteria, and what % of students will be bringing a lunch from home?

24. A rat is placed in a maze set up as in the illustration below:

    When changing chambers (numbered as chambers 1, 2, and 3 in the image), the rat will choose a doorway at random from among those it can see. For example, if the rat is in Chamber 1, it can see five doorways, three of which will lead into Chamber 2, and two of which leads into Chamber 3. That means that there is a $3/5$ probability that the next Chamber that the rat is to occupy will be Chamber 2, and a $2/5$ probability that it will occupy Chamber 3 next.

    (a) Determine the transition matrix P associated with a rat's movement from one chamber to another.

    (b) If a rat is currently in Chamber 2, what is the probability that it will be in Chamber 1, Chamber 2, and Chamber 3, respectively, after changing chamber twice?

    (c) What is the long-term probability that the rat will be in Chamber 1?

25. History has shown that a student who asks for an extension on a homework assignment in Linear Algebra is 40% likely to also need an extension on the next homework assignment. Meanwhile, a student who does not ask for an extension on their current homework assignment is only 10% likely to ask for an extension on their next homework assignment. If there is a 75% chance that Sophie will be able to complete Assignment 4 on time, what is the probability that she will need an extension on Assignment 6?

26. There is a 45% probability that any given child will grow up to become a doctor if their father is a doctor. Meanwhile, a child whose father is *not* a doctor is only 5% likely to become a doctor when they grow up. Dr. Peter Cummings has many sons, but no daughters. What is the probability that his first grandchild will grow up to become a doctor?

27. Although it isn't true that all cats have nine lives, something strange has been happening to felines across Canada. A virus swept through the feline population in January 2015 and, since then, the rules of life and death haven't applied as reliably as in previous years. On any given month after January 2015, a living cat will have a 90% chance of also being alive the next month. However, since the mysterious virus hit, it has been noted that a cat that is dead on one month will have a 98% chance of remaining dead next month, but also a 2% chance of spontaneously coming back to life (assuming that the cat died after the virus had struck).

    (a) Determine the transition matrix P associated with this situation.

    (b) If Mr. Whiskers (your neighbour's cat) is alive today, what is the probability that Mr. Whiskers will be dead two months from now?

    (c) Assuming that no new cats are born, what percentage of the cat population from January 2015 is expected to be alive, and what percentage of the cat population is expected to be dead in the distant future?

Cryptography

28. Working with mod(26) calculations:

    (a) Explain why 7 is considered to be the multiplicative inverse of 15 in the cotext of mod(26) calculations.

    (b) Use matrix multiplication to determine which of the following matrices (B or C) is an inverse matrix mod(26) of the matrix $A = \begin{bmatrix} 5 & 9 \\ 1 & 4 \end{bmatrix}$. (In other words, which of the two matrices below would operate as a decryption matrix for the encoding matrix A?)

$$B = \begin{bmatrix} 24 & 11 \\ 7 & 17 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 3 \\ 9 & 12 \end{bmatrix}$$

29. Encode the word **JELLY** using a Hill 2-cipher with the encoding matrix $A = \begin{bmatrix} 2 & 7 \\ 3 & 10 \end{bmatrix}$.

30. Below are four encoding matrices. In each case, find the decryption matrix.

    (a) $A = \begin{bmatrix} 3 & 2 \\ 4 & 11 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix}$

    (c) $A = \begin{bmatrix} 7 & 3 \\ 4 & 3 \end{bmatrix}$

    (d) $A = \begin{bmatrix} 9 & 2 \\ 5 & 3 \end{bmatrix}$

31. Using the encryption matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$, decode the ciphertext below to reveal a request:

XSQXXRAJYPMC

32. Using the encryption matrix $A = \begin{bmatrix} 7 & 5 \\ 2 & 15 \end{bmatrix}$, decode the ciphertext below to answer the riddle, "What's the difference between 0 and 8?"

QFQHFB

33. Using the encryption matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, decode the ciphertext below to reveal the plea:

WOHZBY

34. Using the encryption matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, decode the ciphertext below to reveal the title of a great movie:

REREHXDW

35. Using the encryption matrix $A = \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}$, decode the ciphertext below to reveal a scientific feat:

RRBSOIMJBLWRIUBW

36. Answers to the following trivia questions are provided in encypted form, each time using one of the three following encryption matrices:

$$A = \begin{bmatrix} 4 & 9 \\ 5 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 5 \\ 5 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$$

    Decrypt the answers to each of the questions below using the encryption matrices mentioned in each case.

    (a) What is the longest river in Europe?
    ODGNMQ, encrypted using A

    (b) Who wrote *The Catcher in the Rye*?
    NVUVRMTISG, encrypted using C

    (c) What word describes a dozen dozen?
    HQWQMK, encrypted using A

    (d) Who won the Nobel Prize in Litterature in 2016?
    IKFLBAXY, encrypted using B

    (e) In 2016, where was is the world's tallest freestanding stucture located?
    YKQQUD, encrypted using B

    (f) Who painted *Luncheon of the Boating Party*?
    FGZGMK, encrypted using C

    (g) In physics, what is defined as the measure of a rotational force on an object?
    GTQHYI, encrypted using A

    (h) What is the average gestation period (in months) for an African elephant?
    AAVSKOAAIX, encrypted using B

37. A spy arrives in a foreign country and, in a train station locker, finds and opens a message that reads

*Go to **UXGHDZHKWY** and introduce yourself as **IDHKXEBL**. You contact will meet you there and identify himself by using the phrase **QQZLWDUMQKGI** into his first sentence. He will give you further instructions.*

Before leaving your previous contact, you were asked to memorize the following three encoding matrices:

$$A_1 = \begin{bmatrix} 2 & 9 \\ 1 & 8 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 10 & 1 \\ 11 & 3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 4 & 1 \\ 7 & 7 \end{bmatrix}$$

Use the encoding matrices, in order, to decode the message from the locker.

(a) What three decryption matrices will you use?

(b) Where must you go?

(c) What name should you use to introduce yourself?

(d) What phrase will your contact use to identify himself?

38. You have a hard time memorizing dates and names and, during a test on 19^{th} century America, you can't seem to remember who was President of the Unites States in 1867. Luckily, you're quite good at math, and so is your best friend, who tosses you her answer in the following note, confident that your history teacher doesn't understand the Hill 2-cipher and would ignore it:

UTIBTODT *Use* $A = \begin{bmatrix} 8 & 9 \\ 3 & 4 \end{bmatrix}$

Unfortunately, your best friend is also pretty bad at remembering names and dates, and she forgets that the correct answer is Andrew Johnson. What (incorrect) answer was in the note?

39. Sally and Steve have been married for ten years and love to leave each other notes written in the form of a Hill 2-cipher. This morning, Sally left the following note for Steve:

DCUHGORKKHWVXUBC

Encoding matrix $A = \begin{bmatrix} 3 & 7 \\ 3 & 12 \end{bmatrix}$

What was the message?

ANSWERS:

1. (a) \$160,000 of energy and \$70,000 of material should be produced.

   (b) Yes, the economy is productive since $(I - C)^{-1} \geq 0$.

   (c) The economy consumes \$158,000 of energy and \$69,000 of material.

2. (a) \$19,000 of Iron and \$10,000 of Steel should be produced.

   (b) Only iron is profitable.

3. (a) The economy should produce \$39000 in goods and \$31000 in services.

   (b) Yes, the economy is productive since $(I - C)^{-1} \geq 0$.

4. (a) \$2100 in services and \$2600 in manufacturing should be produced.

   (b) \$1200 in services and \$1100 in manufacturing is consumed internally.

   (c) Both industries are profitable.

5. (a) Both industries are profitable, and the economy is productive.

   (b) Only the second industry is profitable, and the economy is productive.

   (c) Only the second industry is profitable, and the economy is NOT productive.

6. (a) $\det(I - C) = .2$

   (b) $(I - C)^{-1} = \frac{1}{.2} \begin{bmatrix} .4 & .2 & .2 \\ .1 & .4 & .1 \\ .2 & .2 & .4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ .5 & 2 & .5 \\ 1 & 1 & 2 \end{bmatrix}$

   (c) \$5300 in computing, \$2900 in statistical services and \$6100 in engineering should be produced.

7. Min $z = -12$ at $(12/7, 12/7)$. Other corners: $(0, 0), (4, 0)$.

8. Max $z = -7$ at $(8, 1)$. Other corners: $(9, 0), (13, 0), (12, 2)$.

9. Max $z = 52$ at $(8, 1)$. Other corners: $(1, 1), (4, 8), (7, 8)$.

10. Min $z = 60/17$ at $(28/17, 12/17)$.
    Other corners: $(2, 0), (4, 0), (7, 3)$.

11. Min $z = 0$ occurs at $(12, 12)$. Other corners: $(8, 4), (16, 0)$.
    Note: the region is unbounded, but lines of constancy show that the minimum still exists.

12. Min $z = -4$ occurs on the line segment between $(40/11, 12/11)$ and $(6, 8/3)$. Other corner: $(0, 2)$.
    Note: the region is unbounded, but lines of constancy show that the minimum still exists.

13. Let x =# of dry martinis, y =# of medium martinis.

$$\begin{array}{lrcrcl} \textbf{Maximize } z = & 8x & + & 10y & & \\ \text{subject to} & \frac{5}{3}x & + & \frac{3}{2}y & \leq & 60 \\ & \frac{1}{3}x & + & \frac{1}{2}y & \leq & 16 \\ & x \geq 0, & & y \geq 0 & & \end{array}$$

Maximum Revenue is \$344 when 18 dry martinis and 20 medium martinis are sold. Other corner points: $(0,0), (36,0), (0,32)$.

14. (a) Let x be the number of Freaking Strong coffee bags, and let y be the number Relax Mild coffee bags prepared.

$$\begin{array}{lrcrcl} \textbf{Minimize } z = & 3x & + & 4y & & \\ \text{subject to} & 100x & + & 100y & \geq & 6000 \\ & 300x & + & 100y & \geq & 10000 \\ & 100x & + & 300y & \geq & 10000 \\ & x \geq 0, & & y \geq 0 & & \end{array}$$

(b) Minimum cost = \$200 for 40 bags of Freaking Strong and 20 bags of Relax Mild.
Other corner points: $(100,0), (20,40), (0,100)$.

(c) Weekly purchase: 6 kg Brazilian, 14 kg Colombian and 10 kg Ethiopian.

15. Let x be the number of 21-seat buses, and let y be the number 33-seat buses.

$$\begin{array}{lrcrcl} \textbf{Minimize } z = 300x & + & 400y & & & \\ \text{subject to} & x & + & 3y & \leq & 12 \\ & 20x & + & 30y & \geq & 180 \\ & x \geq 0, & & y \geq 0 & & \end{array}$$

(each 21-seat bus can accommodate 20 students and each 33-seat bus 30 students because of the accompanying adult(s)) Minimum cost = \$2600 for six 21-seat buses and two 33-seat buses.
Other corner points: $(12,0), (9,0)$.

16. (a) Let x be the number of kitchen shelves for the first model, and let y be the number of kitchen shelves for the second model.

$$\begin{array}{lrcrcl} \textbf{Maximize } z = 80x & + & 60y & & & \\ \text{subject to} & 2x & + & 3y & \leq & 240 \\ & 3x & + & 2y & \leq & 210 \\ & 3x & + & y & \leq & 180 \\ & x \geq 0, & & y \geq 0 & & \end{array}$$

Maximum profit = \$6000 at corner (30,60).
Other corner points: $(0,0), (0,80), (50,30), (70,0)$.

(b) Maximum profit = \$6300 at corner (30,60) and at corner (50,30), therefore at every point on the line segment between these two points. There are nine other actual solutions to the problem (whole numbers).

17. (a) Let x be the number of litres of EnviroShine, and let y be the number of litres of Nature's Scent cleaners.

$$\begin{array}{lrcrcl} \textbf{Maximize } z = 1.5x & + & 1.2y & & & \\ \text{subject to} & 0.4x & + & 0.5y & \leq & 94 \\ & 0.3x & + & 0.2y & \leq & 51 \\ & 0.3x & + & 0.3y & \leq & 60 \\ & x \geq 0, & & y \geq 0 & & \end{array}$$

Maximum profit = \$273 at corner (110,90).
Other corner points: $(0,0), (0,188), (60,140), (170,0)$.

(b) 89 L of I_1, 51 L of I_2 and 60 L of I_3.

(c) New constraint: $x \leq 70$. New production: $(70,130)$ with a profit of \$261.
Other corner points: $(0,0), (0,188), (60,140), (70,0)$.

(d) The weekly purchasing must be modified to 93 L of I_1, 47 L of I_2 and 60 L of I_3.

18. (a) $\mathbf{x_3} = \begin{bmatrix} 72/125 \\ 53/125 \end{bmatrix}$

(b) $\mathbf{x_3} = \begin{bmatrix} 9/25 \\ 16/25 \end{bmatrix}$

(c) $\mathbf{x_3} = \begin{bmatrix} 7/20 \\ 27/50 \\ 11/100 \end{bmatrix}$

19. Steady state vectors:

(a) $\begin{bmatrix} 9/16 \\ 7/16 \end{bmatrix}$

(b) $\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

(c) $\begin{bmatrix} 11/48 \\ 9/16 \\ 5/24 \end{bmatrix}$

20. (a) $\begin{bmatrix} 23/54 & 31/72 \\ 31/54 & 41/72 \end{bmatrix}$

(b) There's a $31/54$ chance (or about 57%) that my cat will be asleep three hours from now.

21. Approximately 79% at work and 21% absent.

22. Approximately 71% in the urban core and 29% in the suburbs.

23. Approximately 46% will be buying lunch in the cafeteria, and 54% will be bringing a lunch from home.

24. (a) $P = \begin{bmatrix} 0 & 3/4 & 2/3 \\ 3/5 & 0 & 1/3 \\ 2/5 & 1/4 & 0 \end{bmatrix}$

(b) There is a $1/6$ probability that the rat will be in Chamber 1, a $8/15$ probability that it will be in Chamber 2, and a $3/10$ probability that it will be in Chamber 3.

(c) There is a $5/12$ probability that the rat will be in Chamber 1 long-term. (Note that this means that, over the course of an extended period of time, the rat will wnd up in Chamber 1 roughly $5/12$ of the time.)

25. There is a $^{61}/_{400}$ (15.25%) chance that Sophie will need an extension on Assignment 6.

26. There is a 23% chance that Dr. Cummings' first grandchild will grow up to become a doctor.

27. (a) $P = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$

    (b) There is a 18.8% chance that Mr. Whiskers will be dead two months from now.

    (c) At any given point in the distant future, $^1/_6$ of the cat population from January 2015 would be alive, and $^5/_6$ would be dead.

28. (a) 7 and 15 are multiplicative inverses $\bmod(26)$ because $(7)(15) = 105 \equiv 1 \bmod(26)$.

    (b) B is the inverse of A (and C is not) because

$$AB = \begin{bmatrix} 183 & 208 \\ 52 & 131 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod(26)$$

$$BA = \begin{bmatrix} 131 & 260 \\ 52 & 131 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod(26)$$

$$\text{but } AC = \begin{bmatrix} 121 & 123 \\ 40 & 51 \end{bmatrix} \equiv \begin{bmatrix} 17 & 19 \\ 14 & 25 \end{bmatrix} \bmod(26)$$

29. **CBDZQM**

30. (a) $A^{-1} = \begin{bmatrix} 5 & 2 \\ 4 & 23 \end{bmatrix}$

    (b) $A^{-1} = \begin{bmatrix} 9 & 17 \\ 17 & 4 \end{bmatrix}$

    (c) $A^{-1} = \begin{bmatrix} 9 & 17 \\ 14 & 21 \end{bmatrix}$

    (d) $A^{-1} = \begin{bmatrix} 17 & 6 \\ 15 & 25 \end{bmatrix}$

31. PASS THE SUGAR

32. A BELT

33. HELP ME

34. MEMENTO

35. THEY SPLIT THE ATOM

36. Decryption matrices: $A^{-1} = \begin{bmatrix} 4 & 23 \\ 7 & 10 \end{bmatrix}$

    $B^{-1} = \begin{bmatrix} 5 & 15 \\ 15 & 8 \end{bmatrix}$ $\quad$ $C^{-1} = \begin{bmatrix} 5 & 2 \\ 7 & 17 \end{bmatrix}$

    (a) VOLGA

    (b) J.D. SALINGER

    (c) GROSS

    (d) BOB DYLAN

    (e) RENOIR

    (f) DUBAI

    (g) TORQUE

    (h) TWENTY-TWO

37. (a) $A_1^{-1} = \begin{bmatrix} 16 & 21 \\ 11 & 4 \end{bmatrix}$ $\quad$ $A_2^{-1} = \begin{bmatrix} 7 & 15 \\ 9 & 6 \end{bmatrix}$

    $A_3^{-1} = \begin{bmatrix} 9 & 21 \\ 17 & 20 \end{bmatrix}$

    (b) HOTEL RUBIO

    (c) SAM HILL

    (d) PERFECT STORM

38. LINCOLN $\quad$ (using $A^{-1} = \begin{bmatrix} 6 & 19 \\ 15 & 12 \end{bmatrix}$)

39. GET MILK. I LOVE YOU. $\quad$ (using $A^{-1} = \begin{bmatrix} 6 & 3 \\ 5 & 21 \end{bmatrix}$)