1. Given $A = \begin{bmatrix} 0 & 5 & -1 \\ 3 & 2 & 1 \\ -4 & 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 & 0 & 8 \\ 8 & 3 & -3 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 9 & 1 \\ 1 & -2 \\ 5 & 5 \end{bmatrix}$; find the following if defined.

   (a) AD (b) B^3 (c) D^2 (d) CC^T (e) $A - 2I$ (f) B^{-1}

2. Given $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 6 & 1 \\ -5 & 0 \\ 2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 2 \\ -7 & 6 \end{bmatrix}$; find the following if defined.

   (a) AD (b) AC (c) $A^T A - 2B$ (d) $D^{-1}DD^{-1}$ (e) B^{-1}

3. Given $B = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 9 & -1 \\ 4 & 0 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 4 & -4 \\ 0 & 0 \\ -5 & 2 \end{bmatrix}$; find the following if defined.

   (a) DB (b) BD (c) $CD - B^2$ (d) $(2B)^{-1}$

4. Given $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -2 \\ 1 & 2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}$; find the following if defined.

   (a) AB (b) $2A^T - 3B$ (c) $C^2 - 5I$ (d) $\det(C^7)$

5. Suppose $\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix}$. Find a and b.

6. Suppose $A = \begin{bmatrix} 2 & b \\ c & 3 \end{bmatrix}$. Find b and c so that $A^2 = \begin{bmatrix} 1 & 5 \\ -15 & 6 \end{bmatrix}$.

7. Suppose $A = \begin{bmatrix} a & 0 & -1 \\ 2 & 3 & b \end{bmatrix}$. Find a and b so that $AA^T = \begin{bmatrix} 26 & -11 \\ -11 & 14 \end{bmatrix}$.

8. Suppose $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 8 & 13 & 18 \\ 14 & 19 & 24 \end{bmatrix}$. Find $a, b, c, d, e,$ and f.

9. Find s and t such that A is a symmetric matrix:

   (a) $A = \begin{bmatrix} 1 & s \\ -2 & t \end{bmatrix}$; (b) $A = \begin{bmatrix} s & t \\ st & 1 \end{bmatrix}$; (c) $A = \begin{bmatrix} s & 2s & st \\ t & -1 & s \\ t & s^2 & s \end{bmatrix}$; (d) $A = \begin{bmatrix} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{bmatrix}$; (e) $A = \begin{bmatrix} s & s^2 \\ st & t \end{bmatrix}$.

10. Let A, B and X be $n \times n$ matrices. Solve for X in the following matrix equations (all necessary matrices are assumed to be invertible). Simplify your answers as much as possible.

    (a) $B^{-1}XB = AB$

    (b) $A^{-1}X^{-1} = BA^{-1}$

    (c) $B^T X^T = B + I$

    (d) $A^{-1}(B + X)^{-1} = A^{-1}$

    (e) $ABXA^{-1}B^{-1} = I + A$

11. Given that $A = \begin{bmatrix} 5 & 7 & -2 \\ 0 & 3 & 1 \\ 4 & 0 & 6 \end{bmatrix}$,

(a) find $A_{2,3}$, the (2,3)-cofactor of A;

(b) find $\det(A)$;

(c) find $\operatorname{adj}(A)$;

(d) find A^{-1};

(e) use A^{-1} to solve the system:
$$\begin{cases} 5x + 7y - 2z = -3 \\ \quad\;\; 3y + z = 0 \\ 4x \qquad\;\; + 6z = 2 \end{cases}$$

12. Given that $A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 3 \\ 5 & 6 & 0 \end{bmatrix}$,

(a) find $A_{1,2}$, the $(1,2)$-cofactor of A;

(b) find $\det(A)$;

(c) find $\operatorname{adj}(A)$;

(d) find A^{-1};

(e) use A^{-1} to solve the system:
$$\begin{cases} 2x \qquad\; + 4z = 3 \\ x - y + 3z = 1 \\ 5x + 6y \qquad = 4 \end{cases}$$

13. Given $A = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 3 & -1 \\ -2 & 7 & 1 \end{bmatrix}$, find the following:

(a) $M_{2,3}$, the $(2,3)$-minor of A

(b) $\det(A)$

(c) $\operatorname{adj}(A)$

(d) $A \operatorname{adj}(A)$

(e) $(\operatorname{adj}(A))^{-1}$ (Hint: Look for a quick way.)

14. Consider the system $\begin{cases} 5x_1 \qquad\;\; + x_3 = 2 \\ 2x_1 + 3x_2 - x_3 = 0 \\ -3x_1 + x_2 \qquad = -1 \end{cases}$

(a) Write the system in the form $AX = B$.

(b) Find $\det(A)$, where A is as in the previous part.

(c) Solve for x_1 using Cramer's rule.

(d) Find $\operatorname{adj}(A)$.

(e) Find $A \operatorname{adj}(A)$.

(f) Find A^{-1}.

(g) Solve the system using A^{-1}.

15. Evaluate: $\begin{vmatrix} 2 & 0 & -3 & 3 \\ -3 & 5 & 3 & 0 \\ 4 & 0 & 2 & 1 \\ 0 & 2 & 2 & 5 \end{vmatrix}$

16. Use Cramer's rule to solve:
$$\begin{cases} 7x_1 \qquad\quad + x_3 = 1 \\ 9x_1 + x_2 + x_3 = 1 \\ 8x_1 - 3x_2 \qquad = 0 \end{cases}$$

17. Use Cramer's rule to solve for x_3 in the system:
$$\begin{cases} x_1 - x_2 - x_3 + x_4 = 0 \\ x_1 + 3x_2 + 5x_3 + 4x_4 = -1 \\ x_1 + 3x_2 + 5x_3 + 6x_4 = -1 \\ \qquad 3x_2 + 3x_3 + x_4 = 0 \end{cases}$$

18. Use Cramer's rule to solve for x_3 in the system:
$$\begin{cases} 3x_1 + x_2 + 2x_3 - x_4 = 0 \\ 4x_1 \qquad\quad + 2x_3 + 2x_4 = 0 \\ -x_1 - 2x_2 + 5x_3 + 7x_4 = 1 \\ 2x_1 + 3x_2 + x_3 \qquad = 0 \end{cases}$$

19. Use Cramer's rule to solve for x_4 in the system:
$$\begin{cases} 3x_1 + 4x_2 + x_3 \qquad = 1 \\ 2x_1 - x_2 \qquad\quad + 6x_4 = 1 \\ 5x_1 \qquad\quad + 2x_3 + 5x_4 = 1 \\ -2x_1 - 5x_2 - x_3 \qquad = 1 \end{cases}$$

20. Given that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $\det(A) = 5$, evaluate:

(a) $\det(3A)$

(b) $\det(A^4)$

(c) $\begin{vmatrix} 2a+4b & 2c+4d \\ a & c \end{vmatrix}$

21. Given that $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has $\det(A) = -3$, evaluate:

(a) $\det(2A)$

(b) $\det(A^3)$

(c) $\det((A^{-1})^T)$

(d) $\begin{vmatrix} g & h & i \\ d & e & f \\ 5a-3d & 5b-3e & 5c-3f \end{vmatrix}$

22. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and suppose that $\det(A) = 7$.

Suppose also that B is 3×3 and $\det(B) = 5$. Find the following:

(a) $\det(5AB^{-1})$

(b) $\begin{vmatrix} b & a & c \\ 2e & 2d & 2f \\ h & g & i \end{vmatrix}$

(c) $\begin{vmatrix} 2a & 3b & 5a \\ 2d & 3e & 5d \\ 2g & 3h & 5g \end{vmatrix}$

(d) $\begin{vmatrix} 2a+3d & 2b+3e & 2c+3f \\ d & e & f \\ 5g-7d & 5h-7e & 5i-7f \end{vmatrix}$

23. Suppose A, B, and C are 5×5 matrices such that $\det(A) = 5,\ \det(B) = 6$, and C is not invertible.
Find the following or state that there is not enough information.

(a) $\det(A + B)$

(b) $\det(AC + BC)$

(c) $\det(AC + CB)$

(d) $\det(B^{-1} + B^{-1})$

ANSWERS:

1. (a) $AD = \begin{bmatrix} 0 & -15 \\ 34 & 4 \\ -20 & -6 \end{bmatrix}$

(b) $B^3 = \begin{bmatrix} 95 & -66 \\ 22 & -15 \end{bmatrix}$

(c) D^2 is undefined.

(d) $CC^T = \begin{bmatrix} 69 & -2 \\ -2 & 82 \end{bmatrix}$

(e) $A - 2I = \begin{bmatrix} -2 & 5 & -1 \\ 3 & 0 & 1 \\ -4 & 6 & 0 \end{bmatrix}$

(f) $B^{-1} = \begin{bmatrix} 0 & 1 \\ -1/3 & 5/3 \end{bmatrix}$

2. (a) AD is undefined.

(b) $AC = \begin{bmatrix} -5 & 3 \\ 25 & 5 \end{bmatrix}$

(c) $A^TA - 2B = \begin{bmatrix} 23 & 13 & -6 \\ 11 & 8 & -1 \\ -10 & -5 & 1 \end{bmatrix}$

(d) $D^{-1}DD^{-1} = D^{-1} = \frac{1}{32}\begin{bmatrix} 6 & -2 \\ 7 & 3 \end{bmatrix}$

(e) $B^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 4 & 8 & -3 \\ 4 & 7 & -3 \end{bmatrix}$

3. (a) $DB = \begin{bmatrix} 28 & -4 \\ 0 & 0 \\ -26 & -1 \end{bmatrix}$

(b) BD is undefined.

(c) $CD - B^2 = \begin{bmatrix} -4 & -12 \\ 9 & -7 \end{bmatrix}$

(d) $(2B)^{-1} = \dfrac{1}{22}\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

4. (a) $AB = \begin{bmatrix} 0 & 10 & -4 \\ 1 & -3 & 4 \\ 3 & 31 & -4 \end{bmatrix}$

(b) $2A^T - 3B = \begin{bmatrix} 4 & -17 & 16 \\ -3 & -4 & 0 \end{bmatrix}$

(c) $C^2 - 5I = \begin{bmatrix} 101 & 60 \\ 60 & 29 \end{bmatrix}$

(d) $\det(C^7) = [\det(C)]^7 = 128$

5. $a = 2,\ b = 3$.

6. $b = 1, c = -3$.

7. $a = -5,\ b = 1$.

8. $a = 1,\ b = 2,\ c = 3,\ d = 4,\ e = 5,\ f = 6$.

9. (a) $s = -2, t \in \mathbb{R}$ (t can be any real number)

(b) $s = 1, t \in \mathbb{R} \rightarrow \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}$ **or** $t = 0, s \in \mathbb{R} \rightarrow \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$

(c) $s = 0, t = 0$ **or** $s = 1, t = 2$

(d) $s = 0, t = 3$

(e) $s = 0, t \in \mathbb{R}$ **or** $s = t \in \mathbb{R}$

10. (a) $X = BA$

(b) $X = AB^{-1}A^{-1}$

(c) $X = (B+I)^T B^{-1}$

(d) $X = I - B$

(e) $X = (AB)^{-1}BA + A$

11. (a) $A_{2,3} = 28$

(b) $\det(A) = 142$.

(c) $\operatorname{adj}(A) = \begin{bmatrix} 18 & -42 & 13 \\ 4 & 38 & -5 \\ -12 & 28 & 15 \end{bmatrix}$

(d) $A^{-1} = \dfrac{1}{142}\begin{bmatrix} 18 & -42 & 13 \\ 4 & 38 & -5 \\ -12 & 28 & 15 \end{bmatrix}$

(e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -14/71 \\ -11/71 \\ 33/71 \end{bmatrix}$

12. (a) $A_{1,2} = 15$.

(b) $\det(A) = 8$.

(c) $\operatorname{adj}(A) = \begin{bmatrix} -18 & 24 & 4 \\ 15 & -20 & -2 \\ 11 & -12 & -2 \end{bmatrix}$

(d) $A^{-1} = \dfrac{1}{8}\begin{bmatrix} -18 & 24 & 4 \\ 15 & -20 & -2 \\ 11 & -12 & -2 \end{bmatrix}$

(e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -7/4 \\ 17/8 \\ 13/8 \end{bmatrix}$

13. (a) $M_{2,3} = 40$

(b) $\det(A) = 58$

(c) $\operatorname{adj}(A) = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 6 & 6 \\ 6 & -40 & 18 \end{bmatrix}$

(d) $A\operatorname{adj}(A) = \begin{bmatrix} 58 & 0 & 0 \\ 0 & 58 & 0 \\ 0 & 0 & 58 \end{bmatrix}$

(e) $(\operatorname{adj}(A))^{-1} = \dfrac{1}{\det(A)}A = \dfrac{1}{58}\begin{bmatrix} 6 & -1 & 0 \\ 0 & 3 & -1 \\ -2 & 7 & 1 \end{bmatrix}$

14. (a) $\begin{bmatrix} 5 & 0 & 1 \\ 2 & 3 & -1 \\ -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

(b) $\det(A) = 16$

(c) $x_1 = 5/16$

(d) $\operatorname{adj}(A) = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 3 & 7 \\ 11 & -5 & 15 \end{bmatrix}$

(e) $A\operatorname{adj}(A) = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

(f) $A^{-1} = \dfrac{1}{16}\begin{bmatrix} 1 & 1 & -3 \\ 3 & 3 & 7 \\ 11 & -5 & 15 \end{bmatrix}$

(g) $(5/16, -1/16, 7, 16)$

15. 386

16. $(0, 0, 1)$

17. $x_3 = -\frac{1}{2}$

18. $x_3 = \frac{13}{92}$

19. $x_4 = -\frac{26}{59}$

20. (a) 45

(b) 625

(c) -20

21. (a) -24

(b) -27

(c) $-1/3$

(d) 15

22. (a) 175

(b) -14

(c) 0

(d) 70

23. (a) Not enough information.

(b) 0.

(c) Not enough information. For example, if $A = B = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\det(AC + CB) = x^2$.

(d) $16/3$