1. Given $A = \begin{bmatrix} 3 & 4 & 8 & 0 & 5 & 3 \\ 3 & 2 & -2 & 0 & 2 & 13 \\ 0 & -2 & -10 & 0 & -3 & 10 \\ 6 & 4 & -4 & 0 & 9 & 16 \end{bmatrix} \sim R = \begin{bmatrix} 1 & 0 & -4 & 0 & 0 & 7 \\ 0 & 1 & 5 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Write the last column of A as a linear combination of the first five columns of A.

(b) Find a basis for $\text{Col}(A)$.

(c) Find a basis for $\text{Row}(A)$.

(d) Find a basis for $\text{Nul}(A)$.

(e) What is the dimension of $\text{Nul}(A^T)$?

2. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first performs the horizontal shear that sends the vector $\mathbf{e}_2$ to $-\mathbf{e}_1 + \mathbf{e}_2$ (leaving $\mathbf{e}_1$ unchanged), then reflects points through the y-axis, and then rotates points about the origin by $\frac{\pi}{3}$ radians in the *clockwise* direction. Find the standard matrix for T.

3. Given $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = -3$. Evaluate the following.

(a) $\det\left(\begin{bmatrix} a/5 & b/5 & c/5 \\ g & h & i \\ a+4d & b+4e & c+4f \end{bmatrix}\right)$

(b) $\det(\, 2A^2A^T \,)$

(c) $\det\left(A^{-1} + \text{adj}(A)\right)$

4. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 4 & 6 & 1 \end{bmatrix}$.

(a) Perform exactly one elementary row operation on A to get a *symmetric* matrix. (Call it B)

(Reminder: Matrix B is symmetric if $B = B^T$)

(b) Write B as the product, EA, where E is an elementary matrix.

5. Given X, B, and C are $n \times n$ matrices, solve the following equation for X. Assume any necessary matrices to be invertible.

$$(3X^{-1}B)^{-1} = C(X + B)$$

6. Let $A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 6 & 10 \\ -1 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

(a) Determine A^{-1}.

(b) Use your answer in part (a) to solve the matrix equation $A\mathbf{x} = \mathbf{b}$.

7. Determine a specific 3×4 matrix A that meets the following two conditions. Show the conditions are satisfied.

- The dimension of $\text{Nul}(A)$ is equal to the rank of A.
- $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \text{Nul}(A)$

8. Given $\mathcal{S} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a^2 = d^2 \right\}$.

(a) Does $\mathcal{S}$ contain the 2×2 zero matrix?

(b) Is $\mathcal{S}$ closed under vector addition? Justify.

(c) Is $\mathcal{S}$ closed under scalar multiplication? Justify.

9. Given $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u}$. Use specific vectors to show T is not a linear transformation.

10. Find the point on the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ closest to the point $(1, -3, -10)$.

11. Find a non-zero vector which is orthogonal to the plane that contains the points $P(5, 2, 3)$, $Q(6, 0, 3)$, and $R(7, 5, 1)$.

12. Let $\mathcal{P}_1$ be the plane with equation $2x - y + z = 3$ and let $\mathcal{P}_2$ be the plane with equation $x + y + 2z = 1$.

(a) Find a parametric vector equation for the line of intersection of planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

(b) For what values a and b does the point $(11, a, b)$ lie on this line of intersection?

(c) Find an equation for the plane $\mathcal{P}_3$ that is parallel to $\mathcal{P}_2$ and passes through $(1, 2, 3)$.

(d) Find the distance between $\mathcal{P}_2$ and $\mathcal{P}_3$.

13. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

(a) Find the volume of the parallelepiped formed by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

(b) True or false: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a basis for $\mathbb{R}^3$.

(c) Use your answer in part (a) to find the determinant of the matrix $\begin{bmatrix} 0 & 0 & 0 & 3 \\ 2 & 2 & 0 & 0 \\ -4 & 1 & 3 & 0 \\ 8 & 3 & 1 & 0 \end{bmatrix}$.

14. Fill in the blank with the word *must, might*, or *cannot*, as appropriate.

(a) A system of linear equations with more equations than variables ________________ have a unique solution.

(b) If $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{u}, \mathbf{w}\}$ are both linearly independent sets, then $\{\mathbf{v}, \mathbf{w}\}$ ________________ be linearly independent.

(c) If A is a 3×3 matrix such that $A^3 = I$, then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ ________________ form a basis for $\mathrm{Col}(A)$.

(d) If $\mathbf{u} \cdot \mathbf{v} \neq 0$, then vectors $\mathbf{u}$ and $\mathbf{v}$ ________________ be parallel.

Answers

1. (a) $\begin{bmatrix} 3 \\ 13 \\ 10 \\ 16 \end{bmatrix} = 7\begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 8 \\ -2 \\ -10 \\ -4 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 2 \\ -3 \\ 9 \end{bmatrix}$ (b) $\left\{\begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -3 \\ 9 \end{bmatrix}\right\}$

(c) $\{[1 \;\; 0 \;\; -4 \;\; 0 \;\; 0 \;\; 7], [0 \;\; 1 \;\; 5 \;\; 0 \;\; 0 \;\; -2], [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; -2]\}$ (other answers exist)

(d) $\left\{\begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right\}$ (e) 1

2. $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1+\sqrt{3} \\ \sqrt{3} & 1-\sqrt{3} \end{bmatrix}$

3. (a) $\frac{12}{5}$ (b) -216 (c) $\frac{8}{3}$

4. (a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 4 & 6 & 1 \end{bmatrix} \underset{\sim}{-2R_1 + R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 6 & 1 \end{bmatrix} = B$ (b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 4 & 6 & 1 \end{bmatrix}$

5. $X = 3(B^{-1} - 3C)^{-1}CB$

6. (a) $A^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1/2 & 1 \\ -1 & 0 & -1 \end{bmatrix}$ (b) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1/2 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -5 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (other answers exist)

8. (a) Yes (b) $\mathcal{S}$ is not closed under addition (c) $\mathcal{S}$ is closed under scalar multiplication

9. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\mathrm{T}(\mathbf{u}) = 0$, $\mathrm{T}(\mathbf{v}) = 0 \;\rightarrow\; \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v}) = 0$ however $\mathrm{T}(\mathbf{u} + \mathbf{v}) = 1$

10. $(3, -6, -6)$

11. $(4, 2, 7)$

12. (a) $\mathbf{x} = \begin{bmatrix} 4/3 \\ -1/3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (b) $a = 28/3$, $b = -29/3$ (c) $x + y + 2z = 9$ (d) $\frac{4}{3}\sqrt{6}$

13. (a) 40 u^3 (b) True (c) -120

14. (a) might (b) might (c) must (d) might