1. (6 points) Let $R = \begin{bmatrix} 1 & 3 & 0 & -2 & -1 \\ 0 & 0 & 1 & -8 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ be the reduced row echelon form of the matrix

$A = \begin{bmatrix} 4 & 12 & -2 & 8 & -16 \\ -2 & -6 & 1 & -4 & 8 \\ 5 & 15 & 1 & -18 & 1 \end{bmatrix}$.

(a) Find two distinct bases for $\text{Col}(A)$.

(b) Find a basis for $\text{Row}(A)$.

(c) Find a basis for $\text{Nul}(A)$.

(d) Find $\dim(\text{Nul}(A^T))$.

2. (5 points) Use linear algebra to balance the following chemical equation:

$$\_\_\_\_\_\_ \; CO + \; \_\_\_\_\_\_ \; O_2 \rightarrow \; \_\_\_\_\_\_ \; CO_2$$

3. (5 points) Find all possible combinations of values for h and k that would allow the system of linear equations below to have

(a) infinitely many solutions.

(b) no solution.

(c) a unique solution.

$$\begin{cases} x & & + & 2z & = & 1 \\ -x & + \; ky & + & 6z & = & 3h \\ & 2y & + & 4kz & = & 2 \end{cases}$$

4. (5 points) Consider the matrix $A = \begin{bmatrix} 0 & 5 & 10 \\ -1 & 3 & 11 \end{bmatrix}$.

(a) Find R, the reduced row echelon form of the matrix A.

(b) Express A as a product of elementary matrices multiplied with the matrix R from part (a).

5. (4 points) Give an LU-factorization for the matrix $A = \begin{bmatrix} -3 & 1 \\ 6 & 3 \\ -21 & 27 \end{bmatrix}$.

6. (2 points) Prove that if $\mathbf{u}$ and $\mathbf{v}$ are both solutions to the same matrix equation $A\mathbf{x} = \mathbf{b}$ (where $\mathbf{b} \neq \mathbf{0}$), then the vector $\mathbf{u} + \mathbf{v}$ cannot also be a solution to $A\mathbf{x} = \mathbf{b}$.

7. (10 points) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, let B be another 3×3 matrix, and let L be a 3×3 unit lower triangular matrix. If $\det(A) = 5$ and $\det(B) = -6$, find the values of the following determinants.

(a) If $C = \begin{bmatrix} g & h & i \\ 3a + 2d & 3b + 2e & 3c + 2f \\ a & b & c \end{bmatrix}$, find $\det(C)$.

 Question 7 continues on the next page.

(b) If $C = \begin{bmatrix} g & h & i \\ 3a+2d & 3b+2e & 3c+2f \\ a & b & c \end{bmatrix}$, find $\det(A+C)$.

(c) Find $\det \begin{bmatrix} I & 0 \\ B^T & I+L \end{bmatrix}$.

(d) Find $\det(\text{adj}(B))$.

8. (4 points)

(a) Set up a system needed to express the vector $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$. Do not row reduce.

(b) Use Cramer's Rule to solve your system from part (a). Give the linear combination described as your final answer.

9. (3 points) Solve for X in the matrix equation $2X^T + A = (XB - C)^T$.
Assume that A, B, C are all $n \times n$ matrices and that you will not encounter any non-invertible matrices during your work.

10. (7 points) Let $\mathcal{L}$ be the line $\mathbf{x} = \begin{bmatrix} 2 \\ k \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}$ and let $\mathcal{P}$ be the plane $x + ky - 10z = 5$.

(a) For what value(s) of k, if any, does the point $(-4, 6, -17)$ lie on the line $\mathcal{L}$?

(b) For what value(s) of k, if any, is the line $\mathcal{L}$ parallel to the plane $\mathcal{P}$?

(c) For what value(s) of k, if any, is the line $\mathcal{L}$ orthogonal to the plane $\mathcal{P}$?

(d) For what value(s) of k, if any, does the plane $\mathcal{P}$ intersect the plane $4x - 12y - 40z = 12$ in a line?

11. (8 points) Consider the points $A(3, 5, 0)$, $B(5, 5, -2)$, and $C(3, 8, 2)$.

(a) Find the area of a parallelogram with three of its vertices at the points A, B, and C.

(b) Find an equation of the form $ax + by + cz = d$ for the plane on which the points A, B, and C lie.

(c) Find the distance from the point C to the line through the points A and B.

12. (6 points) Consider the matrix $A = \begin{bmatrix} 6 & 2 & k & 0 \\ 3 & 1 & -2 & 0 \\ k^2 & -2 & 3 & k \\ -12 & k & 8 & 0 \end{bmatrix}$.

(a) Find an expression in terms of k for $\det(A)$.

(b) For what value(s) of k, if any, would $\text{rank}(A) = 4$?

13. (4 points) Consider the subspace $\mathcal{H} = \left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathbb{M}_{2\times 2} \ : \ \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \right\}$.

(a) Find a basis for $\mathcal{H}$.

(b) Give $\dim(\mathcal{H})$.

14. (6 points) Consider the set $\mathcal{R} = \{p(x) \in \mathbb{P}_2 \ : \ p(1)p(0) = 0\}$.
(To be clear: $\mathcal{R}$ is a set of quadratic polynomials for which $p(1)$ multiplied by $p(0)$ equals zero.)

(a) Provide two nonzero polynomials from $\mathcal{R}$, netiher of which is a scalar multiple of the other.

(b) Is the set $\mathcal{R}$ closed under addition? Justify your answer.

(c) Is the set $\mathcal{R}$ closed under scalar multiplication? Justify your answer.

15. (7 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$.

(a) Find the standard matrix A of the transformation T.

(b) Give a nonzero vector from the kernel of T.

(c) Is T onto? Justify your answer.

(d) Find a matrix B of rank 1 such that if $S(\mathbf{x}) = B\mathbf{x}$ then the standard matrix of $S \circ T$ is a zero matrix.

16. (3 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & x \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y \\ 1 \end{bmatrix}$.
Is the transformation T linear? Justify your answer.

17. (3 points) Let the vector $\mathbf{v} \times \mathbf{w}$ have a magnitude of 4 and form an angle of $\frac{\pi}{3}$ with the unit vector $\mathbf{u}$. Use the properties of dot product and of cross product to solve for k in the expression below:

$$\mathbf{u} \cdot (\mathbf{v} \times k\mathbf{w}) + 3\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = -16$$

18. (2 points) Consider the standard matrix A of a rotation transformation about the origin in $\mathbb{R}^2$. Use determinants to show that A must be an invertible matrix, regardless of the angle of rotation θ.

19. (5 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three nonzero vectors in $\mathbb{R}^3$ for which $5\mathbf{u} + 8\mathbf{v} + 3\mathbf{w} = \mathbf{0}$.

(a) Is it possible that $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$? Justify your answer briefly.

(b) If we also know that $\mathbf{u}$ is parallel to $\mathbf{v}$, what is the dimension of $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?

(c) If A is the matrix $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$, find a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

20. (5 points) Complete the following sentences with the word **MUST**, **MIGHT**, or **CANNOT**, as appropriate.

(a) If the first column of a $m \times n$ matrix B is from $\text{Nul}(A)$, then the columns of AB ____________ form a linearly dependent set.

(b) The partitioned matrix $\begin{bmatrix} I & A \\ A & I \end{bmatrix}$ ____________ be symmetric.

(c) If A is a 4×3 matrix and $A\mathbf{x} = \mathbf{b}$ has no solution, then the dimension of the null space of A ____________ be zero.

(d) If A, B, and C are three distrinct points in $\mathbb{R}^3$ such that $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$, then there ____________ exist a single line on which all three points lie.

 Question 20 continues on the next page.

(e) If $\{\mathbf{u}, \mathbf{v}\}$ is a basis for a subspace S, then the set $\{6\mathbf{u} + 3\mathbf{v},\ 10\mathbf{u} + 5\mathbf{v}\}$ ____________ also be a basis for S.

ANSWERS:

1. (a) $\left\{\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 12 \\ -6 \\ 15 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -18 \end{bmatrix}\right\}$ (multiple answers possible)
(b) $\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -8 \\ 6 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}\right\}$ (d) 1
2. $2\,CO + 1\,O_2 \to 2\,CO_2$
3. (a) $k = 2$ and $h = \frac{1}{3}$, or $k = -2$ and $h = -1$ (b) $k = 2$ and $h \neq \frac{1}{3}$, or $k = -2$ and $h \neq -1$
(c) $k \neq \pm 2$ (h can be any real value)
4. (a) $R = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} R$ (multiple answers possible)
5. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & 4 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}$ 6. $A(\mathbf{u}+\mathbf{v}) = 2\mathbf{b} \neq \mathbf{b}$
7. (a) -10 (b) 0 (c) 8 (d) 36
8. (a) $\left[\begin{array}{cc|c} 3 & 7 & 5 \\ -4 & -3 & 8 \end{array}\right]$ (b) $\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \frac{-71}{19}\begin{bmatrix} 3 \\ -4 \end{bmatrix} + \frac{44}{19}\begin{bmatrix} 7 \\ -3 \end{bmatrix}$
9. $X = [(2I - B^T)^{-1}(-A - C^T)]^T$ or $X = -(A^T + C)(2I - B)^{-1}$
10. (a) $k = 8$ (b) $k = \frac{1}{3}$ (c) no such k-value exists (d) $k \neq -3$
11. (a) $2\sqrt{22}$ units^2 (b) $6x - 4y + 6z = -2$ (c) $\sqrt{11}$ units
12. (a) $-3k^3 - 24k^2 - 48k$ (b) $k \neq 0, -4$
13. $\left\{\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -9 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ (multiple answers possible) (b) 3
14. (a) $3x^2 + 5x - 8$ and $5x^2 - x$ (multiple answers possible) (b) No. (c) Yes.
15. (a) $A = \begin{bmatrix} -3 & -1 & -2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$ (multiple answers possible) (c) No.
(d) $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (multiple answers possible)
16. No.
17. $k = -5$ 18. $\det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$
19. (a) No. (b) 1 (c) $\mathbf{x} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$
20. (a) MUST (b) MIGHT (c) MIGHT (d) MUST (e) CANNOT