[Marks]

1. Evaluate the following integrals.

(5) (a) $\displaystyle\int_0^1 \sqrt{2x - x^2}\, dx$

(5) (b) $\displaystyle\int \frac{1}{x^3} e^{\frac{1}{x}}\, dx$

(5) (c) $\displaystyle\int \frac{1}{(x+1)(x^2+1)}\, dx$

(5) (d) $\displaystyle\int \frac{\tan x}{\ln(\cos x)}\, dx$

(5) (e) $\displaystyle\int x \tan^2 x\, dx$

(5) (f) $\displaystyle\int (\arcsin x)^2\, dx$

2. Evaluate the following improper integrals.

(4) (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\arctan x}{1+x^2}\, dx$

(4) (b) $\displaystyle\int_0^{\frac{\pi^2}{16}} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$

3. Find the following limits

(4) (a) $\displaystyle\lim_{x \to 0} (1+3x)^{2 \csc x}$

(4) (b) $\displaystyle\lim_{x \to 0} \frac{x \ln(1+x)}{1 - \cos x}$

(4) 4. Solve the ordinary differential equation $x + 3y^2\sqrt{x^2+1}\,y' = 0$ with the initial condition $y(0) = 1$.

(5) 5. Find the area enclosed by $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi$.

(5) 6. Sketch and shade the region $\mathcal{R}$ enclosed by $y = x^3$, $y = \sqrt{x}$ between $x = 0$ and $x = \frac{1}{2}$.

(a) Set up but **do not evaluate** the integral for the volume of the solid obtained by rotating $\mathcal{R}$ around the line $y = 2$.

(b) Set up but **do not evaluate** the integral for the volume of the solid obtained by rotating $\mathcal{R}$ around the line $x = -2$.

(4) 7. Find the arc length of the curve $y = \ln(1 - x^2)$ in the interval $0 \le x \le \frac{1}{2}$.

(4) 8. Determine the convergence or divergence of the sequence $\{(-1)^n n e^{-n}\}$. Justify your answer.

(4) 9. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

(9) 10. Determine whether the following series converge or diverge. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

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(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+2}\right)^{\frac{n}{2}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$

(8) 11. Determine if the following series converge absolutely or converge conditionally or diverge. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{(2n)!}$

(4) 12. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n} x^{2n}$.

(4) 13. Find the Taylor series of $f(x) = e^{3x+1}$ at $x = 2$.

(3) 14. Suppose there is a positive sequence $\{a_n\}_{n=1}^{\infty}$ which is decreasing and $\lim_{n\to\infty} a_n = 2$. Prove the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{a_n}{n}$ converges.

This is the end of the examination.