[Marks]

1. Evaluate the following integrals.

(5) (a) $\int_0^1 \sqrt{2x - x^2}\, dx$

**Solution:**

$$\int_0^1 \sqrt{2x-x^2}\,dx \overset{\text{completing the square}}{=\!=\!=} \int_0^1 \sqrt{-(x-1)^2+1}\,dx \overset{u=x-1}{\underset{du=dx}{=\!=}} \int_{-1}^0 \sqrt{1-u^2}\,du$$

$$\overset{u=\sin\theta}{\underset{du=\cos\theta\, d\theta}{=\!=}} \int_{-\frac{\pi}{2}}^0 \cos^2\theta\, d\theta = \int_{-\frac{\pi}{2}}^0 \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^0 d\theta + \frac{1}{2}\int_{-\frac{\pi}{2}}^0 \cos 2\theta\, d\theta$$

$$= \frac{1}{2}\theta\Big|_{-\frac{\pi}{2}}^0 + \left(\frac{1}{4}\sin 2\theta\right)\Big|_{-\frac{\pi}{2}}^0 = \frac{\pi}{4}$$

(5) (b) $\int \frac{1}{x^3} e^{\frac{1}{x}}\, dx$

**Solution:**

$$\int \frac{1}{x^3}e^{\frac{1}{x}}\,dx \overset{u=\frac{1}{x}}{\underset{du=-\frac{1}{x^2}dx}{=\!=}} \int -ue^u\,du = \int -u\,de^u$$

$$= -ue^u - \int e^u d(-u) = -ue^u + \int e^u\,du = -ue^u + e^u + C$$

$$= -\frac{1}{x}e^{\frac{1}{x}} + e^{\frac{1}{x}} + C$$

(5) (c) $\int \frac{1}{(x+1)(x^2+1)}\, dx$

**Solution:** Using partial fractions, we have

$$\frac{1}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}$$

So

$$\int \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{2}\int\frac{1}{x+1}\,dx - \frac{1}{2}\int\frac{x}{x^2+1}\,dx + \frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\arctan x + C$$

(5) (d) $\int \frac{\tan x}{\ln(\cos x)}\, dx$

Question 1 continues on the next page.

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**Solution:**

$$\int \frac{\tan x}{\ln(\cos x)}\,dx = \int \frac{\frac{\sin x}{\cos x}}{\ln(\cos x)}\,dx \xlongequal[\substack{du=-\sin x\,dx\\ \sin x\,dx=-du}]{u=\cos x} \int \frac{-du}{u\ln u} \xlongequal[dt=\frac{1}{u}\,du]{t=\ln u} -\int \frac{1}{t}\,dt$$
$$= -\ln|t| + C = -\ln|\ln(\cos x)| + C$$

(5) (e) $\int x\tan^2 x\,dx$

**Solution:**

$$\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx = \int x\,d\tan x - \frac{1}{2}x^2$$
$$\xlongequal{\text{integration by parts}} x\tan x - \int \tan x\,dx - \frac{1}{2}x^2$$
$$= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

(5) (f) $\int (\arcsin x)^2\,dx$

**Solution:**

$$\int (\arcsin x)^2\,dx \xlongequal[x=\sin u]{u=\arcsin x} [dx = \cos u\,du] \int u^2\cos u\,du = \int u^2\,d\sin u$$
$$= u^2\sin u - \int 2u\sin u\,du = u^2\sin u + 2\int u\,d\cos u$$
$$= u^2\sin u + 2\left(u\cos u - \int \cos u\,du\right)$$
$$= u^2\sin u + 2\,(u\cos u - \sin u) + C$$
$$= x\arcsin^2 x + 2(\arcsin x\cos(\arcsin x) - x) + C$$

2. Evaluate the following improper integrals.

(4) (a) $\int_{-\infty}^{\infty} \frac{\arctan x}{1+x^2}\,dx$

**Solution:**

$$\int \frac{\arctan x}{1+x^2}\,dx \xlongequal[du=\frac{1}{1+x^2}\,dx]{u=\arctan x} \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\arctan^2 x + C$$

Question 2 continues on the next page.

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Hence

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\arctan x}{1+x^2}\,dx &= \int_{-\infty}^{0}\frac{\arctan x}{1+x^2}\,dx+\int_{0}^{\infty}\frac{\arctan x}{1+x^2}\,dx\\ &= \lim_{s\to-\infty}\int_{s}^{0}\frac{\arctan x}{1+x^2}\,dx+\lim_{t\to\infty}\int_{0}^{\infty}\frac{\arctan x}{1+x^2}\,dx\\ &= \lim_{s\to-\infty}-\frac{1}{2}\arctan^2 s+\lim_{s\to\infty}\frac{1}{2}\arctan^2 t\\ &= -\frac{1}{2}\left(-\frac{\pi}{2}\right)^2+\frac{1}{2}\left(\frac{\pi}{2}\right)^2=0\end{aligned}$$

(4) (b) $\displaystyle\int_0^{\frac{\pi^2}{16}}\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

**Solution:**

$$\begin{aligned}\int\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx \underset{\substack{du=\frac{1}{2\sqrt{x}}\,dx\\ 2du=\frac{1}{\sqrt{x}}\,dx}}{\overset{u=\sqrt{x}}{=\!=\!=\!=}} 2\int\sin u\,du=-2\cos u+C\\ =-2\cos\left(\sqrt{x}\right)+C\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\frac{\pi^2}{16}}\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= \lim_{t\to0^+}\int_t^{\frac{\pi^2}{16}}\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx=\lim_{t\to0^+}\left(-2\cos\frac{\pi}{4}+2\cos(\sqrt{t})\right)\\ &= -\sqrt{2}+2\end{aligned}$$

3. Find the following limits

(4) (a) $\displaystyle\lim_{x\to0}(1+3x)^{2\csc x}$

**Solution:**

$$\lim_{x\to0}(1+3x)^{2\csc x}=\lim_{x\to0}e^{2\csc x\ln(1+3x)}=e^{\lim_{x\to0}2\csc x\ln(1+3x)}$$

while

$$\lim_{x\to0}2\csc x\ln(1+3x)=\lim_{x\to0}\frac{2\ln(1+3x)}{\sin x}\overset{+}{=}\lim_{x\to0}\frac{2\cdot\frac{3}{1+3x}}{\cos x}=6$$

So

$$\lim_{x\to0}(1+3x)^{2\csc x}=e^6$$

(4) (b) $\displaystyle\lim_{x\to0}\frac{x\ln(1+x)}{1-\cos x}$

[Marks]

**Solution:**

$$\frac{x\ln(1+x)}{1-\cos x} \overset{\text{✚}}{=} \lim_{x\to 0} \frac{\ln(1+x)+\frac{x}{1+x}}{\sin x} = \lim_{x\to 0} \frac{\ln(1+x)}{\sin x} + \frac{x}{\sin x}\cdot\frac{1}{1+x}$$

Then

$$\lim_{x\to 0}\frac{\ln(1+x)}{\sin x} \overset{\text{✚}}{=} \lim_{x\to 0}\frac{\frac{1}{1+x}}{\cos x} = 1$$

while

$$\lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1}{1+x} = 1$$

Hence

$$\lim_{x\to 0}\frac{x\ln(1+x)}{1-\cos x} = 2$$

(4) 4. Solve the ordinary differential equation $x + 3y^2\sqrt{x^2+1}y' = 0$ with the initial condition $y(0) = 1$.

**Solution:** One has

$$x + 3y^2\sqrt{x^2+1}\frac{dy}{dx} = 0$$

Hence

$$xdx + 3y^2\sqrt{x^2+1}dy = 0; \quad xdx = -3y^2\sqrt{x^2+1}dy$$
$$\frac{x}{\sqrt{x^2+1}}\,dx = -3y^2\,dy$$
$$\int\frac{x}{\sqrt{x^2+1}}\,dx = \int -3y^2\,dy$$

Here

$$\int\frac{x}{\sqrt{x^2+1}}\,dx \underset{\substack{du=2xdx\\ x\,dx=\frac{1}{2}du}}{\overset{u=x^2+1}{=\!=}} \frac{1}{2}\int\frac{du}{\sqrt{u}} = \sqrt{u}+C = \sqrt{1+x^2}+C$$

and

$$\int -3y^2\,dy = -y^3 + C$$

So

$$y^3 = -\sqrt{1+x^2}+C$$

From the initial condition, we have

$$1 = -\sqrt{1}+C; \quad C = 2$$

Hence

$$y^3 = -\sqrt{1+x^2}+2$$

(5) 5. Find the area enclosed by $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi$.

[Marks]

**Solution:** Between $x = 0$ and $x = \pi$, $y = \sin x$ and $y = \cos x$ has an intersection at $x = \frac{\pi}{4}$. So the area $S$ is

$$S = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \, dx = (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 1 + 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

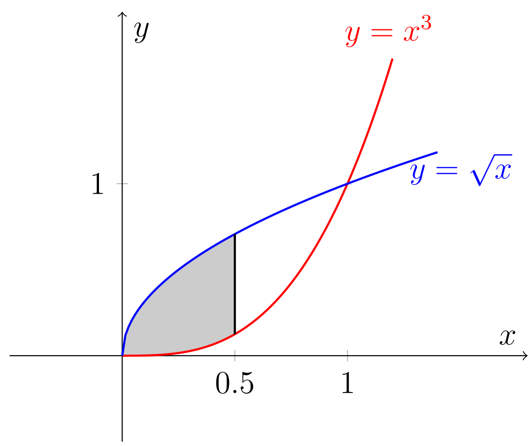
$$= 2\sqrt{2}$$

(5) 6. Sketch and shade the region $\mathcal{R}$ enclosed by $y = x^3$, $y = \sqrt{x}$ between $x = 0$ and $x = \frac{1}{2}$.

(a) Set up but **do not evaluate** the integral for the volume of the solid obtained by rotating $\mathcal{R}$ around the line $y = 2$.

(b) Set up but **do not evaluate** the integral for the volume of the solid obtained by rotating $\mathcal{R}$ around the line $x = -2$.

**Solution:**




(a) $V = \int_0^{\frac{1}{2}} \pi(2 - x^3)^2 - \pi(2 - \sqrt{x})^2 \, dx$

(b) $V = \int_0^{\frac{1}{2}} 2\pi(2 + x)(\sqrt{x} - x^3) \, dx$

(4) 7. Find the arc length of the curve $y = \ln(1 - x^2)$ in the interval $0 \le x \le \frac{1}{2}$.

**Solution:** One has

$$y' = \frac{-2x}{1 - x^2}$$

[Marks]

Hence the length $L$ is

$$L = \int_0^{\frac{1}{2}} \sqrt{1+y'^2}\, dx = \int_0^{\frac{1}{2}} \sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2} = \int_0^{\frac{1}{2}} \sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\, dx$$
$$= \int_0^{\frac{1}{2}} \sqrt{\frac{1+2x^2+x^4}{(1-x^2)^2}}\, dx = \int_0^{\frac{1}{2}} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\, dx = \int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\, dx$$
$$= \int_0^{\frac{1}{2}} \frac{2}{1-x^2} - 1\, dx = -\int_0^{\frac{1}{2}} \frac{2}{x^2-1} + 1\, dx = \left(-\ln\left|\frac{1-x}{1+x}\right| - x\right)\Big|_0^{\frac{1}{2}}$$
$$= \ln 3 - \frac{1}{2}$$

(4) 8. Determine the convergence or divergence of the sequence $\{(-1)^n n e^{-n}\}$. Justify your answer.

**Solution:** Since

$$\lim_{n\to\infty} n e^{-n} = \lim_{n\to\infty} \frac{n}{e^n} \overset{\cdot}{\underset{\cdot}{=}} \lim_{n\to\infty} \frac{1}{e^n} = 0$$

so

$$\lim_{n\to\infty} (-1)^n n e^{-n} = 0$$

Hence the sequence converges to 0.

(4) 9. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

**Solution:** One has

$$\frac{1}{(n+1)(n+3)} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

So

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

For the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$, its partial sum

$$s_n = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

Hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \lim_{n\to\infty} s_n = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

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and consequently,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)} = \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}$$

(9) 10. Determine whether the following series converge or diverge. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

**Solution:** Use the integral test. Take $f(x) = \frac{e^x}{1+e^{2x}}$. Then

$$f'(x) = \frac{e^x(1+e^{2x} - 2e^{3x}}{(1+e^{2x})^2} = \frac{e^x - e^{3x}}{(1+e^{2x})^2} < 0$$

So $f(x)$ is decreasing. Also

$$\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^x}{1+e^{2x}}\,dx = \lim_{t\to\infty} \arctan(e^x)\Big|_1^t = \lim_{t\to\infty} \arctan(e^t) - \arctan(e)$$
$$= \frac{\pi}{2} - \arctan(e)$$

which converges. So from the integral test, the original series converges.

One can also use the ratio test $\left(\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1\right)$ or the root test (L'Hospital's rule is needed and finally one gets $\lim_{n\to\infty} = \frac{1}{e} < 1$). One can also use the limit comparison test where the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$ is used

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+2}\right)^{\frac{n}{2}}$

**Solution:** Use the root test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{3n+2}\right)^{\frac{n}{2}}} = \lim_{n\to\infty} \left(\frac{2n+1}{3n+2}\right)^{\frac{1}{2}} = \lim_{n\to\infty} \sqrt{\frac{2n+1}{3n+2}}$$
$$= \lim_{n\to\infty} \sqrt{\frac{2+\frac{1}{n}}{3+\frac{2}{n}}} = \sqrt{\frac{2}{3}} < 1$$

So the original series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$

[Marks]

**Solution:** Using the limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sin\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \neq 0$$

Hence from the limit comparison test, since $\sum \frac{1}{n^2}$ converges (a $p$-series with $p = 2$), so does the original series.

(8) 11. Determine if the following series converge absolutely or converge conditionally or diverge. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+1}}$

**Solution:** First, we use the limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3+1}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n^3}}} = 1$$

So the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$ diverges because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (a $p$-series with $p = \frac{1}{2}$). Therefore the original series does not converge absolutely.

On the other hand,

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3}}}{\frac{\sqrt{n^3+1}}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n}}}{\sqrt{1+\frac{1}{n^3}}} = 0$$

and

$$\begin{aligned}\left(\frac{n}{\sqrt{n^3+1}}\right)' &= \frac{\sqrt{n^3+1} - n \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{n^3+1}} \cdot 3n^2}{n^3+1} = \frac{n^3+1-\frac{3}{2}n^3}{(n^3+1)\sqrt{n^3+1}} \\ &= \frac{-\frac{1}{2}n^3+1}{(n^3+1)\sqrt{n^3+1}} < 0\end{aligned}$$

when $n$ is big enough, which implies $\frac{n}{\sqrt{n^3+1}}$ decreases. Hence by the alternating series test, the original series converges, and therefore the original series converges conditionally.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{e^n}{(2n)!}$

**Solution:** Use the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(2(n+1))!}}{\frac{e^n}{(2n)!}} = \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \frac{(2n)!}{(2(n+1))!} = \lim_{n\to\infty} \frac{e}{(2n+1)(2n+2)} = 0 < 1$$

[Marks]

Hence from the ratio test, the original series converges absolutely.

(4) 12. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n} x^{2n}$.

**Solution:** Using the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{n+1} x^{2(n+1)}}{\frac{2^n}{n} x^{2n}} \right| = \lim_{n\to\infty} \frac{2n}{n+1} \left| x^2 \right| = 2\left| x^2 \right| < 1$$

Hence $|x^2| < \frac{1}{2}$, so $|x| < \frac{\sqrt{2}}{2}$, that is, the radius of convergence is $\frac{\sqrt{2}}{2}$.

(4) 13. Find the Taylor series of $f(x) = e^{3x+1}$ at $x = 2$.

**Solution:**

$$\begin{aligned} e^{3x+1} &= e^{3(x-2+2)+1} = e^{3(x-2)+7} = e^7 e^{3(x-2)} \\ &= e^7 \sum_{n=0}^{\infty} \frac{(3(x-2))^n}{n!} = \sum_{n=0}^{\infty} \frac{e^7 3^n}{n!} (x-2)^n \end{aligned}$$

(3) 14. Suppose there is a positive sequence $\{a_n\}_{n=1}^{\infty}$ which is decreasing and $\lim_{n\to\infty} a_n = 2$. Prove the series $\sum_{n=1}^{\infty} (-1)^n \frac{a_n}{n}$ converges.

**Solution:** Use the alternating series test. Since the sequence $\{a_n\}$ is positive and decreasing, $\frac{a_n}{n}$ is also positive and decreasing. Also since $\lim_{n\to\infty} a_n = 2$, $\lim_{n\to\infty} \frac{a_n}{n} = 0$. Hence from the alternating series test, $\sum_{n=1}^{\infty} (-1)^n \frac{a_n}{n}$ converges.

This is the end of the examination.