

1. (10 points) Evaluate the following limits. Use  $-\infty$ ,  $\infty$  or “does not exist”, wherever appropriate.

$$(a) \lim_{x \rightarrow 2} \frac{\frac{3}{x+1} - \frac{4}{x^2}}{x-2}$$

$$(b) \lim_{x \rightarrow e^+} \frac{\ln(x) - 3}{1 - \ln(x)}$$

$$(c) \lim_{x \rightarrow 4^+} \frac{|x^2 - 16| - |8 - 2x|}{4 - x}$$

$$(d) \lim_{x \rightarrow 0} \frac{\sin(x) + \sin(3x)}{2x}$$

(e) Use the Squeeze Theorem to evaluate

$$\lim_{x \rightarrow 0^+} \left[ x^{2/3} \cos\left(\frac{2}{3x}\right) - 2 \right]$$

2. (5 points) Find the equations of all vertical and horizontal asymptotes of the graph of

$$f(x) = \frac{3x + 5 + \sqrt{4x^2 + 1}}{2x - \sqrt{x^2 + 27}}$$

3. Consider the piecewise defined function:

$$f(x) = \begin{cases} ax + 5 & \text{if } x < 5 \\ 2^{a+3} - 6 & \text{if } x = 5 \\ \frac{x^2 - 25}{ax - 5a} & \text{if } x > 5 \end{cases}$$

- (a) (2 points) Find all values of  $a$ , if any, for which  $\lim_{x \rightarrow 5} f(x)$  exists.
- (b) (2 points) Find all values of  $a$ , if any, for which  $f$  is continuous at  $x = 5$ .
4. Consider the function  $f(x) = \sqrt{10 - x^2}$ .
- (a) (3 points) Find  $f'(x)$  using the limit definition of the derivative.
- (b) (1 point) Check your answer from part (a) by using the rules of differentiation.
- (c) (2 points) Write an equation of the tangent line at the point where  $x = 1$ .

5. (20 points) Find  $\frac{dy}{dx}$  for each of the following. Do not simplify your answers.

$$(a) y = 5^x + x^5 + \frac{1}{\sqrt{x^5}} + \frac{1}{\sqrt{5}}$$

$$(b) y = e^{5x^2} \cos(\log_2(x))$$

$$(c) y = \frac{(x^3 + 1)^4}{\tan^6(2x) + 10}$$

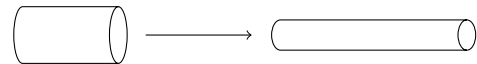
$$(d) y = (x \sec(x))^{x^2}$$

$$(e) x^2 - \cot(xe^y) = 3xy$$

6. (4 points) Use logarithmic differentiation to find the derivative of the function  $y = \frac{3 \csc^4(x)}{x^5 \cdot \sqrt[6]{\ln(x)}}$

7. (3 points) Find the 96<sup>th</sup> derivative,  $f^{(96)}(x)$ , for the function  $f(x) = \frac{x}{7-x}$ .

8. (5 points) An elastic cylindrical tube is being stretched along its axis by pulling on its ends. The object retains a cylindrical shape as it is stretched, and the volume remains constant at  $40\pi \text{ cm}^3$ . At a given instant, the length of the cylinder is observed to be 10 cm but increasing at the rate of 5 cm/min. What is the rate of change of the radius at that instant?

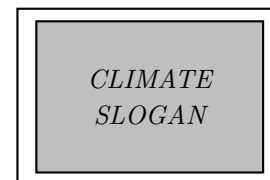


9. (11 points) Consider the following function, along with its first and second derivatives.

$$f(x) = \frac{2(x-3)^3}{27(x-1)} \quad f'(x) = \frac{4x(x-3)^2}{27(x-1)^2}$$

$$f''(x) = \frac{4(x-3)(x^2+3)}{27(x-1)^3}$$

- (a) Find the domain and intercepts of  $f$ .
- (b) Find the vertical and horizontal asymptotes of  $f$  (if any).
- (c) Find the intervals of increase/decrease of  $f$ .
- (d) Find the local (relative) extrema of  $f$ .
- (e) Find the intervals of concavity of  $f$ .
- (f) Find all points of inflection of  $f$ .
- (g) On the next page, sketch a graph of  $f$ .
10. (5 points) You are designing a new rectangular sign for the next climate strike. You need an area of  $24 \text{ m}^2$  for your witty slogan, but you need to leave a blank space of  $1/2 \text{ m}$  wide on each side and  $1/3 \text{ m}$  on the top and bottom of your slogan.
- What should be the dimensions (width and height) of your climate slogan in order to minimize the total area of cardboard needed to make the sign?



11. (4 points) Find the absolute maximum and minimum values of the function  $f(x) = \sqrt[3]{x(x-32)}$  on the interval  $-8 \leq x \leq 1$ .
12. (3 points) A particle moves with a variable acceleration given by  $a(t) = e^t + 4 \sin(t) - 2 \cos(t)$ . Find the position  $s(t)$  of this particle as a function of time if the initial position is  $s(0) = 0$  with an initial velocity of  $v(0) = 2$ .

13. (9 points) Compute each integral below.

$$(a) \int \frac{(\sqrt{x} - 3)(2x - 1)}{\sqrt{x^3}} dx$$

$$(b) \int_2^4 \left[ \left(\frac{2}{y}\right)^2 + e^2 \right] dy$$

$$(c) \int \frac{5 \sin(x) - e^x \cos^2(x)}{\cos^2(x)} dx$$

14. (3 points) Consider the function defined by

$$f(x) = \begin{cases} |x| - 2 & \text{for } -3 \leq x \leq 3, \\ 1 & \text{elsewhere.} \end{cases}$$

Sketch the graph of  $f$  and use it to calculate  $\int_{-5}^5 f(x) dx$ .

15. (3 points) Evaluate the integral  $\int_0^5 (4 - x^2) dx$  by expressing it as a limit of Riemann sums.

You might use the formulas:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

16. (2 points) Given  $F(x) = \int_{e^{2x}}^{\pi} t \tan(t) dt$ , find  $F'(x)$ .

17. (3 points) Use the Intermediate Value Theorem to show that the equation  $x^4 - x^3 - x = 4$  has at least one positive and one negative root (solution).

### Answers

1. (a)  $\frac{2}{3}$

(b)  $\infty$

(c)  $-6$

(d)  $2$

(e)  $-2$

2. vertical asymptote:  $x = 3$ , horizontal asymptotes:  $y = \frac{1}{3}$ ,  $y = 5$

3. (a)  $a = -2$ ,  $a = 1$

(b)  $a = 1$  only

4. (a)  $f'(x) = \lim_{h \rightarrow 0} \frac{-2x - h}{\sqrt{10 - (x+h)^2} + \sqrt{10 - x^2}} = \frac{-x}{\sqrt{10 - x^2}}$

(b)  $f'(x) = \frac{1}{2}(10 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{10 - x^2}}$

(c)  $y = -\frac{1}{3}x + \frac{10}{3}$

5. (a)  $\frac{dy}{dx} = 5^x \ln(5) + 5x^4 - \frac{5}{2}x^{-7/2}$

(b)  $\frac{dy}{dx} = e^{5x^2} \cdot 10x \cdot \cos(\log_2(x)) - e^{5x^2} \cdot \sin(\log_2(x)) \cdot \frac{1}{x \ln(2)}$

(c)  $\frac{dy}{dx} = \frac{4(x^3 + 1)^3 \cdot 3x^2 \cdot (\tan^6(2x) + 10) - (x^3 + 1)^4 \cdot 6 \tan^5(2x) \cdot \sec^2(2x) \cdot 2}{(\tan^6(2x) + 10)^2}$

(d)  $\frac{dy}{dx} = (x \sec(x))^{x^2} [2x \ln(x \sec(x)) + x + x^2 \tan(x)]$

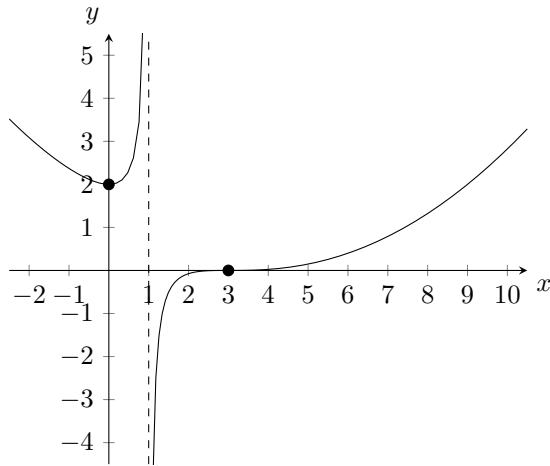
(e)  $\frac{dy}{dx} = \frac{3y - 2x - e^y \csc^2(xe^y)}{xe^y \csc^2(xe^y) - 3x}$

6.  $\frac{dy}{dx} = -\frac{3 \csc^4(x)}{x^5 \cdot \sqrt[6]{\ln(x)}} \cdot \left[ 4 \cot(x) + \frac{5}{x} + \frac{1}{6x \ln(x)} \right]$

7.  $f^{(96)}(x) = \frac{7 \cdot 96!}{(7-x)^{97}}$

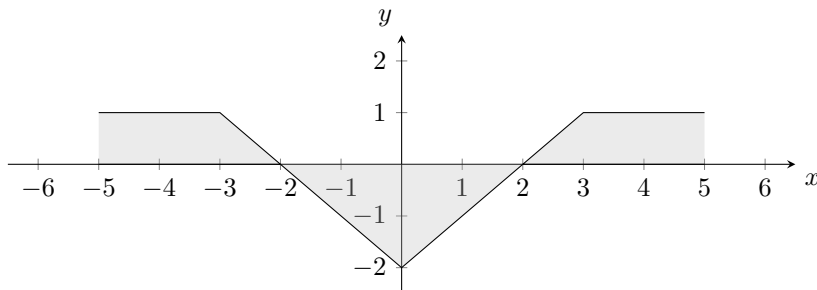
8.  $\frac{dr}{dt} = -\frac{1}{2} \text{ cm/min}$

9. (a) domain:  $\mathbb{R} \setminus \{1\}$   
 $y$ -intercept:  $(0, 2)$   
 $x$ -intercept:  $(3, 0)$
- (b) vertical asymptote:  $x = 1$   
horizontal asymptote: none  $\left( \lim_{x \rightarrow \pm\infty} f(x) = \infty \right)$
- (c) increasing on  $(0, 1) \cup (1, \infty)$   
decreasing on  $(-\infty, 0)$
- (d) local minimum:  $(0, 2)$   
local maximum: none
- (e) concave upward on  $(-\infty, 1) \cup (3, \infty)$   
concave downward on  $(1, 3)$
- (f) inflection point:  $(3, 0)$
- (g)



10. width = 6 m, height = 4 m
11. absolute maximum at  $(-8, 80)$   
absolute minimum at  $(1, -31)$
12.  $s(t) = e^t - 4 \sin(t) + 2 \cos(t) + 5t - 3$
13. (a)  $2x - \ln|x| - 12x^{1/2} - 6x^{-1/2} + C$   
(b)  $1 + 2e^2$   
(c)  $5 \sec(x) - e^x + C$

14.



$$\int_{-5}^5 f(x) dx = 1$$

15.  $-\frac{65}{3}$

16.  $F'(x) = -e^{2x} \tan(e^{2x}) \cdot e^{2x} \cdot 2$

17.  $f(-2) = 22$  and  $f(0) = -4 \Rightarrow$  there is at least one root in  $(-2, 0)$  (a negative root)  
 $f(0) = -4$  and  $f(2) = 2 \Rightarrow$  there is at least one root in  $(0, 2)$  (a positive root)