

1. Solve each of the following systems or show that it is inconsistent.

$$(a) \begin{cases} 5x - 2y - 2z = 13 \\ -2x + y + z = -3 \\ 10x + y + 8z = 67 \end{cases}$$

$$(b) \begin{cases} x_1 + 4x_3 + 3x_4 = -2 \\ 3x_1 + 2x_2 + 15x_4 = 6 \\ 7x_1 + 6x_2 - 8x_3 + 39x_4 = 29 \end{cases}$$

2. Given the matrix representation of a system as $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 1 & 1 & k+3 & -1 \\ 0 & -2 & 10 & k^2 - 2k - 19 \end{array} \right]$

Find the value(s) of k , if any, for which the system has

- (a) no solution
- (b) infinitely many solutions
- (c) a unique solution

3. Given $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$, calculate the following or state that they do not exist.

- (a) A^{-1}
- (b) B^2
- (c) $3A + B$

(d) Find a vector \mathbf{x} such that $B^{-1}\mathbf{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

4. Consider the system of equations $\begin{cases} 2x + 9y + 6z = 0 \\ 2x + 10y + 4z = -1 \\ 4x + 18y + 10z = 0 \end{cases}$

- (a) If A is the coefficient matrix, find A^{-1} .
- (b) Solve the system using A^{-1} .
- (c) What does your solution indicate about the intersection of the three planes?

5. Given $A = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ -1 & 2 \end{bmatrix}$, solve $(AX)^T = B$ for X .

6. What makes an $n \times n$ square matrix A symmetric?

7. Let A , B , and C be 3×3 matrices. It is given that $\text{rank}(A) = 2$, $\det(B) = 3$ and $\det(C) = -2/3$. Find the following, or state that there is not enough information.

- (a) $\det(A)$
- (b) $\det(3C^{-1})$
- (c) $\det(AB)$
- (d) $\det((2B^T)^{-1})$
- (e) $\det(A + B)$

8. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$, find

(a) $\begin{vmatrix} 4a & 4b & 4c \\ d-g & e-h & f-i \\ g-d & h-e & i-f \end{vmatrix}$

(b) $\begin{vmatrix} d & e & f \\ 2a & 2b & 2c \\ \frac{1}{5}g + 3d & \frac{1}{5}h + 3e & \frac{1}{5}i + 3f \end{vmatrix}$

9. Use Cramer's Rule to solve the following linear system $AX = B$ for \mathbf{x}_2 only.

(Do *not* solve for x_1 , x_3 , or x_4 !)

Note that $\det(\mathbf{A}) = -30$.

$$\begin{cases} 2x_1 + x_2 - x_3 + 2x_4 = 5 \\ + 2x_2 + x_3 + 3x_4 = 0 \\ x_1 + x_2 + x_3 - 2x_4 = -4 \\ x_1 - x_2 + x_4 = 4 \end{cases}$$

10. Given the points $P(1, -2, 1)$, $Q(2, -1, 2)$ and $R(2, 3, -1)$

(a) Find the vector \overrightarrow{PQ} and its magnitude.

(b) Find a unit vector parallel to \overrightarrow{PQ} .

(c) Find an equation in vector form of the line passing through the points Q and R .

(d) Does the point $(1, 2, -3)$ lie on the line found in (c)? Justify your answer.

(e) Is the vector \overrightarrow{PQ} parallel to the line in (c)? Justify.

(f) Write a vector equation of the plane passing through the points P, Q and R .

(g) Write an equation in the form $ax + by + cz = d$ of the plane passing through the points P, Q and R .

11. Consider the point $A(1, 5, -2)$ and the plane $P: 3x - y + 2z = 7$.

(a) Is the point A on the plane P ? Justify your answer.

(b) Find a vector equation of the line through the point A and perpendicular to the plane P .

12. Consider the planes $P_1: 4x + 2y + 5z = 11$ and $P_2: 2x - 14y + 4z = -3$. Are P_1 and P_2 parallel, perpendicular, or neither?

13. Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid xy \geq 0 \right\}$

(a) Is the vector $\mathbf{0}$ in S ?

(b) Find two non-zero vectors in S .

(c) Is S a subspace of \mathbb{R}^2 ? If so, provide a basis. If not, provide a counterexample for one of the closure properties.

14. Let A be an $n \times n$ matrix, and suppose that \mathbf{v} is in $\text{Nul}(A)$.
- $A\mathbf{v} = \underline{\hspace{2cm}}$?
 - Is \mathbf{v} in $\text{Nul}(A^2)$? Explain why or why not.
15. Consider the vectors $\mathbf{a}_1 = (1, -2, 3)$, $\mathbf{a}_2 = (3, 1, 1)$ and $\mathbf{a}_3 = (-2, -17, k)$. Find the value(s) of k (if any) so that the vectors
- are linearly dependent
 - are linearly independent
 - span \mathbb{R}^3
16. Label each statement as TRUE or FALSE, and justify your answers.
- If A is 6×8 and the null space of A^T has dimension 2, then the columns of A span a plane.
 - Let A be $n \times n$. If A^{-1} exists, then the null space of A is always $\{\mathbf{0}\}$.

17. Given that the matrix

$$A = \begin{bmatrix} 1 & 3 & 11 & 2 & -1 & -8 & 0 \\ -3 & -8 & -30 & -4 & 2 & 21 & -1 \\ 2 & 4 & 16 & 0 & 1 & -8 & 3 \\ 2 & 5 & 19 & 2 & -3 & \mathbf{k} & -1 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 0 & 2 & -4 & 0 & -3 & 1 \\ 0 & 1 & 3 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Find a basis for $\text{Col}(A)$.
 - What is the dimension of $\text{Col}(A)$?
 - How many vectors are in $\text{Col}(A)$?
 - Find a basis for $\text{Nul}(A)$.
 - What is the dimension of $\text{Nul}(A)$?
 - What is the rank of A^T ?
 - What is the nullity of A^T ?
 - Find the value of k .
18. Solve using the Simplex Method. Your answer should include both the minimum value of C and the corresponding basic feasible solution.
- Minimize** $C = x - 2y - 3z$
 subject to
- $$\begin{array}{rcl} x & & \leq 4 \\ & y & \leq 4 \\ x + & y & \leq 6 \\ -x & & + 2z \leq 4 \\ x, y, z & \geq & 0 \end{array}$$
19. A baker has 60 units of flour, 132 units of sugar, and 102 units of raisins in stock. A loaf of raisin bread requires 1 unit of flour, 1 unit of sugar, and 2 units of raisins, while a raisin cake requires 2, 4, and 1 units of flour, sugar, and raisins, respectively. A loaf of raisin bread sells for \$3 and each raisin cake sells for \$4.

- (a) Define your variables and set up the linear program that would be used to determine the production level needed to maximize revenue. When defining x_1 etc., please respect the order in which things were named in the question. **Do not solve.**

Below is the final simplex table corresponding to this linear program. Use it to answer the questions that follow.

$$\left[\begin{array}{ccc|ccc|c} 1 & 0 & 0 & 5/3 & 0 & 2/3 & 168 \\ \hline 0 & 0 & 1 & 2/3 & 0 & -1/3 & 6 \\ 0 & 0 & 0 & -7/3 & 1 & 2/3 & 60 \\ 0 & 1 & 0 & -1/3 & 0 & 2/3 & 48 \end{array} \right]$$

- (b) How many loaves of raisin bread and how many raisin cakes should be baked to maximize revenue?
 (c) What is the maximum revenue?
 (d) When revenue is maximized, how much flour, sugar, and raisins remain?
20. Secret Agent Indigo just finished a dangerous mission, and received the message GAJYCM from her agency to tell her where to fly to, so she can lay low for a while. The message is encoded with a Hill cipher: $\begin{bmatrix} 3 & 19 \\ 2 & 5 \end{bmatrix}$. Where is she going?

The following tables are provided for reference:

The alphabet:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

Reciprocals mod 26:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

21. Bill wants to use a 2×2 Hill cipher to encode his communications, and randomly came up with the following encoding matrix and shared it with Joan. However when Joan tried to decode the message she was sent, she could not. What was the problem?

$$\begin{bmatrix} 3 & 17 \\ 5 & 13 \end{bmatrix}$$

22. In the town of Jacville, there are only two political parties: the blue and yellow parties. They run for an election every year.

A historical analysis has shown the following: if the yellow party wins an election, then they have a 75% chance of winning the following election as well. If the blue party wins an election, then they have a 60% chance of winning the following election.

- (a) Write down the transition matrix associated with this situation.
 (b) Given that the blue party won this year, what is the probability that the yellow party will win the election in 2 years?
 (c) Find the steady state vector \mathbf{q} . Use fractions rather than decimals.
 (d) In the long run, assuming this trend continues, how often will the blue party be in power?

THE ANSWERS:

1. (a) $x = 7, y = 13, z = -2$
 (b) Inconsistent

2. (a) not possible
 (b) $k = -5$
 (c) $k \neq -5$
3. (a) DNE
 (b) $\begin{bmatrix} 6 & -2 \\ -3 & 7 \end{bmatrix}$
 (c) $\begin{bmatrix} 6 & 14 \\ 12 & 17 \end{bmatrix}$
 (d) $\begin{bmatrix} 10 \\ 4 \end{bmatrix}$
4. (a) $\frac{1}{2} \begin{bmatrix} -14 & -9 & 12 \\ 2 & 2 & -2 \\ 2 & 0 & -1 \end{bmatrix}$
 (b) $x = \frac{9}{2}, y = -1, z = 0$
 (c) It's the single point $(\frac{9}{2}, -1, 0)$.
5. $X = \frac{-1}{11} \begin{bmatrix} 1 & -10 \\ -15 & 7 \end{bmatrix}$
6. $A = A^T$
7. (a) 0 (b) $-\frac{81}{2}$ (c) 0 (d) $\frac{1}{24}$ (e) not enough info
8. (a) 0 (b) $-\frac{8}{5}$
9. $\frac{7}{5}$
10. (a) $\overrightarrow{PQ} = (1, 1, 1); \|\overrightarrow{PQ}\| = \sqrt{3}$
 (b) $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ or $(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})$
 (c) $\mathbf{x} = (2, -1, 2) + t(0, 4, -3)$ (other answers are possible)
 (d) No
 (e) No
 (f) $\mathbf{x} = (1, -2, 1) + s(1, 1, 1) + t(0, 4, -3)$ (other answers are possible)
 (g) $-7x + 3y + 4z = -9$
11. (a) No
 (b) $\mathbf{x} = (1, 5, -2) + t(3, -1, 2)$ (other answers are possible but this is by far the most obvious one)
12. Perpendicular
13. (a) Yes
 (b) Many answers are possible; for instance: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
 (c) Not a subspace; not closed under vector addition.

14. (a) $\mathbf{0}$ (b) Yes, since $A^2\mathbf{v} = A(A\mathbf{v}) = A\mathbf{0} = \mathbf{0}$.

15. (a) $k = 18$ (b) $k \neq 18$ (c) $k \neq 18$

16. (a) FALSE (b) TRUE

17. (a) $\left\{ \left[\begin{array}{c} 1 \\ -3 \\ 2 \\ 2 \end{array} \right], \left[\begin{array}{c} 3 \\ -8 \\ 4 \\ 5 \end{array} \right], \left[\begin{array}{c} -1 \\ 2 \\ 1 \\ -3 \end{array} \right] \right\}$

(b) 3

(c) ∞

(d) $\left\{ \left[\begin{array}{c} -2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 4 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 3 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{array} \right] \right\}$

(e) 4

(f) 3

(g) 1

(h) $k = -17$

18. $C = -15$ @ $(2, 4, 3, 2, 0, 0, 0)$

19. (a) Let $x_1 = \#$ of breads, $x_2 = \#$ of cakes.

$$\begin{aligned} \text{Maximize } R &= 3x_1 + 4x_2 \\ \text{subject to } & x_1 + 2x_2 \leq 60 \\ & x_1 + 4x_2 \leq 132 \\ & 2x_1 + x_2 \leq 102 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

(b) 48 breads and 6 cakes

(c) \$168

(d) No flour, 60 units of sugar, and no raisins remain when revenue is maximized.

20. Newark

21. The matrix is not invertible mod 26.

22. (a) Letting state 1 = yellow and state 2 = blue, $P = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}$

(b) 54%

(c) $\mathbf{q} = \begin{bmatrix} 8/13 \\ 5/13 \end{bmatrix}$

(d) Approximately 38% of the time.