1. (8 points) Given $A = \begin{bmatrix} 2 & -4 & 2 & 2 \\ 3 & -7 & 2 & 2 \\ 4 & -7 & 5 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 0 \\ -9 \\ -1 \end{bmatrix}$.

(a) Find the general solution to the equation $A\mathbf{x} = \mathbf{b}$. Give your answer in parametric vector form.

(b) Find a specific solution to the equation $A\mathbf{x} = \mathbf{b}$ such that $x_1 = x_2$.

(c) Write the third column of A as a linear combination of the first two columns of A or explain why it is not possible to do so.

(d) True or False: There exists a vector $\mathbf{c} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{c}$ does not have a solution.

(e) True or False: The last three columns of A form an invertible matrix.

2. (3 points) The graph of $y = ax^2 + bx + c$ contains the point $(3, 27)$ and $(2, -6)$. The tangent line at $x = 2$ has slope 12. Set up (but do not solve) a linear system for finding a, b, and c.

3. (4 points) Given $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 2 & 3 & -1 \\ 0 & -1 & 8 \end{bmatrix}$

(a) Find A^{-1}

(b) Given that $\det(A) = \frac{1}{2}$, find $\operatorname{adj}(A)$.

4. (3 points) Given that $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 10$, find $\det\left(\begin{bmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{bmatrix}\right)$

5. (3 points) Matrices A and B are invertible. Solve for X in the matrix equation

$$A^{-1}B(X+I)^{-1}A = 2A$$

6. (6 points) You are given the block matrix $B = \begin{bmatrix} O & A \\ I & O \end{bmatrix}$, where A is invertible.

(a) Find B^{-1}

(b) Find B^4

(c) If $A^5 = I$ (but $A \neq I$), find the smallest positive integer m so that $B^m = I$.

7. (7 points) You are given $A = \begin{bmatrix} 3 & 6 & 3 & 21 \\ -2 & -4 & -2 & -14 \\ -1 & -2 & 2 & 2 \end{bmatrix}$ and its reduced row echelon form $R = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for $\operatorname{Col}(A)$.

(b) Find a basis for $\operatorname{Row}(A)$.

(c) Find a basis for $\operatorname{Nul}(A)$

(d) State the dimension of $\operatorname{Nul}(A^T)$.

(e) Circle the answer that correctly completes the sentence.
The column space of A is _____.

1. empty
2. a point
3. a line
4. a plane
5. all of $\mathbb{R}^3$
6. all of $\mathbb{R}^4$

8. (5 points) Let A be an $m \times n$ matrix such that $n > m$.

(a) Explain why $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

(b) What size is the matrix A^TA?

(c) Explain why A^TA cannot be invertible.

9. (4 points) Let $H = \left\{A \in M_{2\times 2} : A^2 = A + A^T\right\}$.

(a) For what values of c is $\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix}$ in H?

(b) Show that H is not a subspace of $M_{2\times 2}$.

10. (4 points) Find a basis for the vector space $V = \{p \in \mathbb{P}_3 : p(1) = p(2)\}$.

11. (3 points) Given each of the following matrices, indicate whether the columns are a linearly independent ("L.I.") or a linearly dependent ("L.D.") set. No justification is necessary.

(a) A 4×5 matrix with a pivot in every row.

(b) The product of two elementary matrices.

(c) The standard matrix of a one-to-one matrix transformation.

12. (12 points) A, B, and C are all $n \times n$ matrices (but n is not provided). You are given the following information.

- $\det(A) = 2$
- $\det(2A) = 32$
- $\det(AB) = 6$
- $\dim(\text{Nul}(AC)) = 1$

Find each of the following:

(a) $\det(A^5A^T)$

(b) $\det(B^{-1})$

(c) n

(d) $\det(-A)$

(e) $\det(BC)$

 Question 12 continues on the next page.

(f) $\operatorname{rank}(AB)$

13. (4 points) Let $A = \begin{bmatrix} 3 & 4 & 1 \\ 9 & 11 & 5 \\ -6 & 13 & 8 \end{bmatrix}$. Find the LU factorization of A.

14. (5 points) You are given the following two lines.

$\mathcal{L}_1$ is $\mathbf{x} = \begin{bmatrix} 0 \\ -4 \\ -2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathcal{L}_2$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Find the cosine of the angle θ formed between $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Is the angle θ between 0 and $\frac{\pi}{3}$?

15. (4 points) Find the point on the line $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ that is nearest to $(-9, 15, -1)$.

16. (3 points) Find the area of the triangle formed by the points $A(4, 2, 1)$, $B(3, 1, 5)$, and $C(2, 3, 6)$

17. (3 points) Find an equation for the line that contains the origin and is parallel to both of the following planes: $x + y + z = 1$, and $2x - y + z = 3$.

18. (6 points) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the y-axis. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical shear transformation such that $S\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

(a) Find the standard matrix for R.

(b) Find the standard matrix for S.

(c) Find the standard matrix for $S^{-1} \circ R$.

19. (2 points) Let Q be a solid object with volume 7.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 4 & k & 0 \\ 2 & 5 & 2 \end{bmatrix} \mathbf{x}$.

Find all k so that the volume of $T(Q)$ is 35.

20. (5 points) Complete each of the following sentences with MUST, MIGHT, or CANNOT.

(a) Let A be a square matrix. If $A\mathbf{x} = A\mathbf{y}$ for distinct $\mathbf{x}$ and $\mathbf{y}$, then A ____________ be invertible.

(b) Let T be a matrix transformation with standard matrix A. The kernel of T ____________ equal the null space of A. The range of T ____________ equal the column space of A.

(c) Let $S = \{\mathbf{u}, \mathbf{v}\}$ be a set of vectors. If $\mathbf{w}$ is in $\operatorname{Span}\{S\}$, then $\mathbf{w}$ ____________ be in S.

(d) Let A be a non-zero, non-invertible 3×3 matrix. If the column space of A does not form a line, then the null space of A ____________ form a line.

21. (4 points) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. The following is given:

- $||\mathbf{u}|| = 3$
- $\mathbf{v}$ is a unit vector
- $\mathbf{u} + 2\mathbf{v}$ is orthogonal to $\mathbf{u} + 3\mathbf{v}$

Find:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $||\mathbf{u} + \mathbf{v}||$

22. (2 points) Prove the following statement.

"If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and $\mathbf{w} \notin \text{Span}\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent."

ANSWERS

1. (a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 9/2 \\ 9/2 \\ -1/2 \\ 5 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ -7 \\ -7 \end{bmatrix}$ (d) FALSE
(e) TRUE

2. $\left[\begin{array}{ccc|c} 9 & 3 & 1 & 27 \\ 4 & 2 & 1 & -6 \\ 4 & 1 & 0 & 12 \end{array}\right]$ 3. (a) $A^{-1} = \begin{bmatrix} 46 & -11 & -10 \\ -32 & 8 & 7 \\ -4 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 23 & -11/2 & -5 \\ -16 & 4 & 7/2 \\ -2 & 1/2 & 1/2 \end{bmatrix}$ 4. 150

5. $\frac{1}{2}A^{-1}B - I$ 6. (a) $\begin{bmatrix} 0 & I \\ A^{-1} & 0 \end{bmatrix}$ (b) $\begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix}$ (c) $m = 10$

7. (a) $\left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ (d) 1 (e) a plane

8. (a) There are more columns than rows in A, so the RREF must contain at least one free variable.
(b) $n \times n$ (c) $(A^TA)\mathbf{x} = \mathbf{0}$ has non-trivial solutions since it is equivalent to $A^T(A\mathbf{x}) = \mathbf{0}$

9. (a) $c = \pm 1$ (b) H is neither closed under scalar multiplication, nor under addition (many counter-examples possible) 10. $\{x^3 - 7x,\ x^2 - 3x,\ 1\}$ (other answers possible)

11. (a) L.D. (b) L.I. (c) L.I. 12. (a) 64 (b) $\frac{1}{3}$ (c) 4 (d) 2 (e) 0 (f) 4

13. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -21 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 52 \end{bmatrix}$ 14. (a) $(3, -1, 4)$ (b) $\frac{1}{6}$ (c) No. 15. $(-11, 9, 5)$

16. $\frac{3\sqrt{11}}{2}$ 17. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ 18. (a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 \\ -7 & 1 \end{bmatrix}$

19. $k = \pm\frac{5}{2}$ 20. (a) CANNOT (b) MUST, MUST (c) MIGHT (d) MUST

21. (a) -3 (b) 2 22. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then there must exist a nontrivial linear combination $k_1\mathbf{u} + k_2\mathbf{v} + k_3\mathbf{w} = \mathbf{0}$ that must have $k_3 \neq 0 \Rightarrow \mathbf{w} = \frac{-k_1}{k_3}\mathbf{u} - \frac{k_2}{k_3}\mathbf{v}$, which contradicts the fact that $\mathbf{w} \notin \text{Span}\{\mathbf{u}, \mathbf{v}\}$.