1. (4 points) Find a polynomial $p(x)$ of degree 2 such that $p(1) = 1$, $p(2) = 6$, and $p'(2) = 6$.

2. (4 points) Consider the following system:

$$\begin{cases} 2x + 3y + 4z = 3 \\ 2x + (h+1)y + 6z = 4 \\ 4x + 6y + (h-4)z = k-1 \end{cases}$$

Find all value(s) of h and k such that the system has:

(a) A unique solution.

(b) Infinitely many solutions.

(c) No solution.

**** Be very clear about your combinations of *h* and *k* in each case ****

3. (10 points) Consider the following matrix A and its reduced row echelon form B given below.

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 3 \\ 3 & 4 & -14 & 1 & 0 & 14 \\ 1 & -1 & 7 & 2 & 0 & -9 \\ -3 & -2 & 4 & 3 & 0 & -24 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & -5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the following sets.

(i) $\text{Col}(A)$

(ii) $\text{Col}(A^T)$

(iii) $\text{Nul}(A)$

(b) Determine the dimension of $\text{Nul}(A^T)$.

(c) Write the fourth column of matrix A as a linear combination of the other columns of A.

(d) Determine a vector that is in $\text{Nul}(A)$ whose first and second entries are 18 and -5, respectively.

(e) Given that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, determine the general solution to $A\mathbf{x} = \mathbf{b}$.

4. (3 points) Let $A = \begin{bmatrix} k & 1 & 2 \\ 1 & k & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find $\det(A)$ in terms of k.

(b) Find all values for k such that A is non-invertible.

5. (6 points) Let S and T both be linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let S be the transformation that reflects vectors through the x-axis. Let T be *a horizontal shear* such that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(a) Find the standard matrix A of transformation S.

(b) Find the standard matrix B of transformation T.

(c) Find the standard matrix C of transformation $S \circ T$.

(d) Find a non-zero vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $S(\mathbf{v}) = T(\mathbf{v})$.

6. (5 points) Let $A = \begin{bmatrix} 2 & 4 \\ 2 & 6 \\ -1 & 6 \\ 0 & -6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -6 \\ -14 \\ -29 \\ 24 \end{bmatrix}$.

(a) Determine an LU- factorization for matrix A.

(b) Use your answer in part (a) to solve the equation $A\mathbf{x} = \mathbf{b}$.

7. (3 points) Express the matrix $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ as a product of elementary matrices.

8. (3 points) Given $A = \begin{bmatrix} -2 & 6 \\ 4 & -7 \end{bmatrix}$, find a matrix X such that $XA - XA^T = A$.

9. (4 points) Let A and B be 4×4 matrices where $\det(A)$ is unknown and $\det(B) = -2$.

   (a) If $A^T A = 2I$. What are the possible values of $\det(A)$?

   (b) Find $\det(\,(B^{-1})^3 \operatorname{adj} B\,)$.

10. (3 points) Let A be a 3×3 matrix such that $A\begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$, and $A\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

    (a) Find a vector $\mathbf{u}$ such that $A\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

    (b) Determine A^{-1}.

11. (2 points) Matrix A is called "skew-symmetric" if $A^T = -A$. Given that $n \times n$ matrices A and B are skew−symmetric and $AB = -BA$, show that matrix AB must also be skew-symmetric.

12. (3 points) Consider a single matrix A with all the following properties:

- A is in RREF.
- The first two columns of A form a linearly independent set.
- The null space of A is in $\mathbb{R}^5$.
- The span of the fourth column of A is a point in $\mathbb{R}^3$.
- The dimension of the null space of A is greater than the rank of A.

    (a) What size matrix must A be?
    (b) Give an example of a matrix that satisfies all the conditions above.

13. (3 points) Let $\begin{bmatrix} W & O \\ X & Y \\ Z & I \end{bmatrix}\begin{bmatrix} A & O \\ B & I \end{bmatrix} = \begin{bmatrix} I & O \\ O & A \\ O & I \end{bmatrix}$. Given that A is a square matrix, express the block entries $W, X, Y,$ and Z in terms of A and B.

    Justify each step.

14. (7 points) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by $T(p(x)) = \begin{bmatrix} p(1) \\ p'(0) \end{bmatrix}$.

    (a) Find the image of $q(x) = x + x^2$ under transformation T.

    (b) Find a polynomial $r(x)$ such that $T(r(x)) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

    (c) Find a basis for the kernel of T.

    (d) Is T a one-to-one transformation? Justify.

15. (6 marks) Let $\mathcal{H} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x(2y - 3z) = 0 \right\}$.

    (a) Is $\mathcal{H}$ closed under vector addition? Justify.

    (b) Is $\mathcal{H}$ closed under scalar multiplication? Justify.

    (c) Is $\mathcal{H}$ a subspace of $\mathbb{R}^3$?

16. (3 points) Let $\mathcal{H} = \left\{ \mathbf{x} \in \mathbb{R}^4 : \mathbf{x} \cdot \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = 0 \right\}$. Given that $\mathcal{H}$ is a subspace of $\mathbb{R}^4$, find a basis for $\mathcal{H}$.

17. (3 points) Let $p(x) = 2 + 3x - x^2$, $q(x) = -3 - 5x + 3x^2$, and $r(x) = 4 + 3x + 7x^2$. Show that

    $r(x) \in \text{Span}\{\, p(x),\ q(x) \,\}$.

18. (8 points) Given the planes $\mathcal{P}_1$: $2x - y + 5z = 8$ and $\mathcal{P}_2$: $x - y + 2z = 4$, and given the point $A(5, -3, 2)$.

    (a) Find the cosine of the angle between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

    (b) Find the point on the plane $\mathcal{P}_1$ closest to A.

(c) Find an equation for the line containing point A that is parallel to both $\mathcal{P}_1$ and $\mathcal{P}_2$.

19. (5 points) Given the line $\mathcal{L}:\ \mathbf{x} = \begin{bmatrix} k \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} h \\ 1 \\ -2 \end{bmatrix}$ and the plane $\mathcal{P}: x - 2y + 4z = 0$, find conditions on h and k for each of the following cases.

**** Be very clear about your combinations of h <u>and</u> k in each case ****

(a) $\mathcal{L}$ and $\mathcal{P}$ are parallel and do not intersect.

(b) $\mathcal{L}$ is contained within plane $\mathcal{P}$.

(c) $\mathcal{L}$ and $\mathcal{P}$ intersect at a point.

20. (9 points) Given the points $P(0,0,-2)$, $Q(2,3,4)$, $R(4,\ 6,\ 5)$, and $S(6,11,10)$, find the following:

(a) The area of triangle PQR.

(b) An equation of the form $ax + by + cz = d$ for the plane containing points P, Q, and R.

(c) The volume of the parallelepiped with edges $\overrightarrow{PQ}, \overrightarrow{PR}$, and $\overrightarrow{PS}$.

(d) A point on the line through P and Q which is two units away from P.

21. (2 points) Show that if $\text{Proj}_{\mathbf{v}}\mathbf{u} = \text{Proj}_{\mathbf{v}}\mathbf{w}$, then $\mathbf{u} - \mathbf{w}$ is orthogonal to $\mathbf{v}$.

22. (4 points) Complete the following sentences with the word *must, might* or, *cannot*, as appropriate.

(a) If $\mathbf{u}$ is in Span$\{\mathbf{v}\}$, then $\mathbf{v}$ ________________ be in Span$\{\mathbf{u}\}$.

(b) If A is invertible, then Row(A) ________________ be identical to Col(A).

(c) If the 3×3 coefficient matrix A of the system of linear equations $A\mathbf{x} = \mathbf{b}$ has rank 2, then the system ________________ be inconsistent when $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) If A, B, and C are square matrices such that $ABC^2 = I$ then matrix C ________________ be invertible.

ANSWERS

1. $p(x) = -2 + 2x + x^2$

2. (a) $h \neq 2$ or $12, k \in \mathbb{R}$ (b) $h = 12$ and $k = 7$ OR $h = 2$ and $k = 2$ (c) $h = 12$ and $k \neq 7$ OR $h = 2$ and $k \neq 2$

3. (a) (i) $\left\{\begin{bmatrix} 2 \\ 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}\right\}$ (ii) $\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}\right\}$ (iii) $\left\{\begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}\right\}$ (b) 1

   (c) $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 3 \\ 1 \\ -3 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 1 \\ 4 \\ -1 \\ -2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 3 \\ 14 \\ -9 \\ -24 \end{bmatrix}$ (d) $\begin{bmatrix} 18 \\ -5 \\ -4 \\ -20 \\ 0 \\ -5 \end{bmatrix}$ (e) $\left\{\begin{array}{l} x_1 = 1 - 2r - 2t \\ x_2 = 2 + 5r - 3t \\ x_3 = 3 + r \\ x_4 = 4 + 4t \\ x_5 = 5 + s \\ x_6 = 6 + t \end{array}\right\}$

4. (a) $\det(A) = k^2 - 3k + 2$ (b) $k = 1$ or 2

5. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (d) $\mathbf{v} = \begin{bmatrix} k \\ 0 \end{bmatrix}; k \in \mathbb{R}$

6. (a) $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1/2 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $L\mathbf{y} = \mathbf{b} \rightarrow \mathbf{y} = \begin{bmatrix} -6 \\ -8 \\ 0 \\ 0 \end{bmatrix}$; $U\mathbf{x} = \mathbf{y} \rightarrow \mathbf{x} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (many answers)

8. $X = A(A - A^T)^{-1} = \begin{bmatrix} 3 & 1 \\ -7/2 & -2 \end{bmatrix}$

9. (a) $\det(A) = \pm 4$ (b) 1

10. (a) $\mathbf{u} = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} 4/3 & -3 & 4 \\ 2 & 0 & 3 \\ 3 & -1 & 1 \end{bmatrix}$

11. $(AB)^T = (-BA)^T = -A^T B^T = -(-A)(-B) = -(AB)$

12. (a) 3×5 (b) $A = \begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $a, b, c, d \in \mathbb{R}$

13. $W = A^{-1}$, $X = -ABA^{-1}$, $Y = A$, $Z = -BA^{-1}$

14. (a) $T\big(q(x)\big) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (b) $r(x) = a + 2x + cx^2$; where $a + c = 3$ (c) $\{1 - x^2\}$ (d) No. Many counter-examples possible.

15. (a) No. Many counter-examples possible. (b) Yes. Given $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \epsilon\, \mathcal{H}$, show $k\mathbf{u} = \begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \end{bmatrix} \epsilon\, \mathcal{H}$ for all k. (c) No, not closed under addition.

16. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

17. $r(x) = 11p(x) + 6q(x)$

18. (a) $\cos\theta = \frac{13}{6\sqrt{5}}$ (b) $(4,\ -5/2,\ -1/2)$ (c) $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

19. (a) $h = 10$ and $k \neq 4$ (b) $h = 10$ and $k = 4$ (c) $h \neq 10$ and $k \in \mathbb{R}$

20. (a) $\frac{1}{2}\sqrt{325}$ (b) $15x - 10y = 0$ (c) 20 units^3 (d) $(4/7,\ 6/7,\ -2/7)$ or $(-4/7,\ -6/7,\ -26/7)$

21. $\dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \ \rightarrow\ \mathbf{u} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v} \ \rightarrow\ \mathbf{u} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} = 0 \ \rightarrow\ (\mathbf{u} - \mathbf{w}) \cdot \mathbf{v} = 0$

22. (a) might (not if $\mathbf{u}$ is the zero vector) (b) must (both equal all $\mathbb{R}^n$) (c) might (as that column changes during row-reduction)

(d) must (and $C^{-1} = ABC$)