

1. (5 points) Consider the following systems of linear equations

$$\begin{cases} x + y - z = 0 \\ x - y + 2z = 0 \\ 3x + y = 0 \end{cases} \quad \text{and} \quad \begin{cases} x + y - z = a \\ x - y + 2z = b \\ 3x + y = c \end{cases}$$

(Note that the two systems above have the same coefficients.)

- (a) Find the general solution to the first system. Give your answer in parametric vector form.
- (b) For some constants a , b , and c , the second system has a particular solution $x = 1, y = 1, z = 1$. Write the general solution for this new system of linear equations in parametric vector form.
2. (2 points) Show that, for any square matrix A and positive integer $n > 1$, all vectors in $\text{Nul}(A)$ must also be in $\text{Nul}(A^n)$.
3. (4 points) Find a polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points $(-1, 6)$, $(1, 24)$, and $(2, 48)$.
4. (2 points) Consider the matrix A , as well as its RREF R below:

$$A = \begin{bmatrix} 4 & 5 & -12 & 3 & 8 \\ 3 & 1 & 2 & 5 & 17 \\ -2 & -1 & 0 & 2 & -5 \\ 5 & 2 & 2 & -1 & 18 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using only the columns of A , give two distinct bases for $\text{Col}(A)$.

5. (6 points) Consider the matrix equation

$$A^{-1}B = (C - 2A)^{-1}$$

- (a) Solve for A in the equation above.
- (b) If $B = \begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 5 & -3 \end{bmatrix}$ in the matrix equation above, evaluate the matrix A .
6. (3 points) Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^n . Show that if $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$, then \mathbf{u} and \mathbf{v} must have the same length.
7. (6 points) Given that A and B denote 4×4 matrices such that that $\det(A^2B) = 20$ and $\det(AB^2) = 50$,
- (a) find $\det(A)$ and $\det(B)$.
- (b) find $\det(A^{-1})$
- (c) find $\det(3B^T)$

8. (4 points) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a nonzero value n . Use Cramer's Rule to solve for x_3 only in the system of linear equations below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3b + 2c \\ 3e + 2f \\ 3h + 2i \end{bmatrix}$$

9. (6 points) For the system of equations:
$$\begin{cases} 2kx + (k+1)y & = & 2 \\ x + y + z & = & 0 \\ -kx + (1-2k)y & = & -1 \end{cases}$$
, find value(s) of k such

that the system has:

- (a) No solution
 (b) One solution
 (c) Infinitely many solutions
10. (2 points) Show that $A^T(4A)$ must be symmetric.

11. (5 points) Let R be the reduced row echelon form of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ -5 & 3 & -7 \end{bmatrix}$.

- (a) Find the RREF matrix R .
 (b) Express A as a product of R with a few elementary matrices.

12. (6 points) Let T be a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \det \begin{bmatrix} 1 & 0 & x \\ 2 & 1 & y \\ 0 & 3 & z \end{bmatrix}$$

- (a) Evaluate $T \left(\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \right)$.

- (b) Find the standard matrix for the linear transformation T .
 (c) Find a basis for $\ker(T)$.

13. (4 points) Find the LU -factorization of the matrix $A = \begin{bmatrix} 2 & -6 & -2 & 4 \\ -1 & 0 & 3 & 2 \\ -1 & 15 & 7 & 10 \end{bmatrix}$

14. (7 points) Let A , B , and C denote matrices with A and C invertible.

- (a) Show that the block matrix $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is invertible by finding an expression for M^{-1} .

- (b) Use the previous result to find the inverse of $M = \begin{bmatrix} 1/2 & 0 & 0 & 1 & 1 \\ 0 & 1/2 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$.

15. (10 points) You are given the following points: $A = (3, 1, 1)$, $B = (2, 1, 3)$, and $C = (1, 0, 3)$.

- (a) Find the distance from the point B to the line through the points A and C .
 (b) Find the point on the line containing A and C that is closest to B .

- (c) Find the cosine of the angle θ formed by \overrightarrow{AB} and \overrightarrow{AC}
- (d) Find the area of the triangle with vertices at points A , B , and C .
16. (3 points) Let \mathcal{L}_1 be the line defined by $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, where $s \in \mathbb{R}$ and \mathcal{L}_2 be the line defined by $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$, where $t \in \mathbb{R}$. Find the normal equation ($ax + by + cz = d$) of the plane that contains \mathcal{L}_1 and is parallel to \mathcal{L}_2 .
17. (4 points) Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y^2 = 4x^2 \right\}$
- (a) Is V closed under vector addition? Justify.
- (b) Is V closed under scalar multiplication? Justify.
18. (5 points) Consider the polynomials $p(x) = 2 + x - x^2$, $q(x) = 3 + 2x + 2x^2$ and $r(x) = 3 + 4x + 16x^2$.
- (a) Show that $r(x)$ is in $\text{Span}\{p(x), q(x)\}$.
- (b) Let \mathbb{P}_2 be the vector space of all polynomials of degree at most 2. Is $\{p(x), q(x), r(x)\}$ a basis for \mathbb{P}_2 ? Justify your answer.
19. (4 points) Find the standard matrix for the combination of linear transformations $S \circ T$ if $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformations which rotates vectors about the origin by $\frac{\pi}{3}$ radians counter-clockwise, and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ projects vectors onto the x -axis.
20. (6 points) Let $\mathcal{H} = \left\{ A \in M_{2 \times 2} : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = A \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.
- (a) Given that \mathcal{H} is a subspace, find a basis for it.
- (b) What is the dimension of \mathcal{H} ?
- (c) Can $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$ be written as a linear combination of your basis vectors? Justify.
21. (6 points) Complete each of the following sentences with MUST, MIGHT, or CANNOT.
- (a) Let \mathbf{u}, \mathbf{v} , and \mathbf{w} be distinct nonzero vectors in \mathbb{R}^3 . If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ _____ be equal to $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$.
- (b) If A is a square matrix and $A^4 - 2A^2 + A = I$, then A _____ be invertible.
- (c) If A and B are $n \times n$ matrices such that $AB = B$, then A _____ be an identity matrix.
- (d) If $A\mathbf{x} = \mathbf{b}$ has two distinct solutions then the columns of A _____ be linearly dependent.
- (e) Let \mathcal{L}_1 and \mathcal{L}_2 be lines in \mathbb{R}^2 , where \mathcal{L}_1 does not pass through the origin and \mathcal{L}_2 passes through the origin. If there exists a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(\mathcal{L}_1) = \mathcal{L}_2$, then T _____ be one-to-one.
- (f) If A is an $m \times n$ matrix and $\text{Nul}(A) = \mathbb{R}^n$, then A _____ be a $m \times n$ zero matrix.

ANSWERS

1. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$
2. $A^n \mathbf{x} = A^{n-1} A \mathbf{x} = A^{n-1} \mathbf{0} = \mathbf{0}$ 3. $p(x) = 10 + 9x + 5x^2$
4. $\left\{ \begin{bmatrix} 4 \\ 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \\ -1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 5 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -12 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \\ -1 \end{bmatrix} \right\}$ (for example)
5. (a) $A = (B^{-1} + 2I)^{-1}C$ or $A = B(I + 2B)^{-1}C$ (depending on approach) (b) $\begin{bmatrix} 3 & 1 \\ -7 & 0 \end{bmatrix}$
6. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0 \Rightarrow \mathbf{u} \cdot \mathbf{u} + 0(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v} \cdot \mathbf{v} = 0 \Rightarrow \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0 \Rightarrow \|\mathbf{u}\| = \|\mathbf{v}\|$
7. (a) $\det(A) = 2, \det(B) = 5$ (b) $\frac{1}{2}$ (c) 405
8. 3 9. (a) $k = 0$ (b) $k \neq 0, 1$ (c) $k = 1$
10. $[A^T(4A)]^T = (4A)^T(A^T)^T = 4A^T A = A^T(4A)$
11. (a) $R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
12. (a) 5 (b) $[6 \ -3 \ 1]$ (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/6 \\ 0 \\ 1 \end{bmatrix} \right\}$ (others possible)
13. $A = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & -2 & 4 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 14 & 28 \end{bmatrix}$
14. (a) $\begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 & -2 & 2 \\ 0 & 2 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & -3 \end{bmatrix}$
15. (a) 1 unit (b) $(\frac{5}{3}, \frac{1}{3}, \frac{7}{3})$ (c) $\frac{2\sqrt{5}}{5}$ (d) $\frac{3}{2}$ units²
16. $-2x - 6y + z = -12$ 17. (a) No (b) Yes 18. (a) $r(x) = -6p(x) + 5q(x)$ (b) No
18. $\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 0 \end{bmatrix}$ 19. (a) $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$ (for example) (b) 2 (c) No
20. (a) CANNOT (b) MUST (c) MIGHT (d) MUST (e) CANNOT (f) MUST