1. (5 points) Consider the following systems of linear equations

$$\left\{\begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ x & - & y & + & 2z & = & 0 \\ 3x & + & y & & & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcrcrcl} x & + & y & - & z & = & a \\ x & - & y & + & 2z & = & b \\ 3x & + & y & & & = & c \end{array}\right.$$

(Note that the two systems above have the same coefficients.)

(a) Find the general solution to the first system. Give your answer in parametric vector form.

(b) For some constants a, b, and c, the second system has a particular solution $x = 1, y = 1, z = 1$. Write the general solution for this new system of linear equations in parametric vector form.

2. (2 points) Show that, for any square matrix A and positive integer $n > 1$, all vectors in $\mathrm{Nul}(A)$ must also be in $\mathrm{Nul}(A^n)$.

3. (4 points) Find a polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points $(-1, 6)$, $(1, 24)$, and $(2, 48)$.

4. (2 points) Consider the matrix A, as well as its RREF R below:

$$A = \begin{bmatrix} 4 & 5 & -12 & 3 & 8 \\ 3 & 1 & 2 & 5 & 17 \\ -2 & -1 & 0 & 2 & -5 \\ 5 & 2 & 2 & -1 & 18 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using only the columns of A, give <u>two</u> distinct bases for $\mathrm{Col}(A)$.

5. (6 points) Consider the matrix equation

$$A^{-1}B = (C - 2A)^{-1}$$

(a) Solve for A in the equation above.

(b) If $B = \begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 5 & -3 \end{bmatrix}$ in the matrix equation above, evaluate the matrix A.

6. (3 points) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that if $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ must have the same length.

7. (6 points) Given that A and B denote 4×4 matrices such that that $\det(A^2B) = 20$ and $\det(AB^2) = 50$,

(a) find $\det(A)$ and $\det(B)$.

(b) find $\det(A^{-1})$

(c) find $\det(3B^T)$

8. (4 points) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a nonzero value n. Use Cramer's Rule to solve for x_3 only in the system of linear equations below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3b + 2c \\ 3e + 2f \\ 3h + 2i \end{bmatrix}$$

9. (6 points) For the system of equations: $\begin{cases} 2kx + (k+1)y = 2 \\ x + y + z = 0 \\ -kx + (1-2k)y = -1 \end{cases}$, find value(s) of k such that the system has:

(a) No solution

(b) One solution

(c) Infinitely many solutions

10. (2 points) Show that $A^T(4A)$ must be symmetric.

11. (5 points) Let R be the reduced row echelon form of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ -5 & 3 & -7 \end{bmatrix}$.

(a) Find the RREF matrix R.

(b) Express A as a product of R with a few elementary matrices.

12. (6 points) Let T be a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \det \begin{bmatrix} 1 & 0 & x \\ 2 & 1 & y \\ 0 & 3 & z \end{bmatrix}$$

(a) Evaluate $T\left(\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}\right)$.

(b) Find the standard matrix for the linear transformation T.

(c) Find a basis for $\ker(T)$.

13. (4 points) Find the LU-factorization of the matrix $A = \begin{bmatrix} 2 & -6 & -2 & 4 \\ -1 & 0 & 3 & 2 \\ -1 & 15 & 7 & 10 \end{bmatrix}$

14. (7 points) Let A, B, and C denote matrices with A and C invertible.

(a) Show that the block matrix $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is invertible by finding an expression for M^{-1}.

(b) Use the previous result to find the inverse of $M = \begin{bmatrix} 1/2 & 0 & 0 & 1 & 1 \\ 0 & 1/2 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$.

15. (10 points) You are given the following points: $A = (3, 1, 1)$, $B = (2, 1, 3)$, and $C = (1, 0, 3)$.

(a) Find the distance from the point B to the line through the points A and C.

(b) Find the point on the line containing A and C that is closest to B.

 Question 15 continues on the next page.

(c) Find the cosine of the angle θ formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(d) Find the area of the triangle with vertices at points A, B, and C.

16. (3 points) Let $\mathcal{L}_1$ be the line defined by $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, where $s \in \mathbb{R}$ and $\mathcal{L}_2$ be the line defined by $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$, where $t \in \mathbb{R}$. Find the normal equation $(ax + by + cz = d)$ of the plane that contains $\mathcal{L}_1$ and is parallel to $\mathcal{L}_2$.

17. (4 points) Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y^2 = 4x^2 \right\}$

(a) Is V closed under vector addition? Justify.

(b) Is V closed under scalar multiplication? Justify.

18. (5 points) Consider the polynomials $p(x) = 2 + x - x^2$, $q(x) = 3 + 2x + 2x^2$ and $r(x) = 3 + 4x + 16x^2$.

(a) Show that $r(x)$ is in $\text{Span}\{p(x), q(x)\}$.

(b) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2. Is $\{p(x), q(x), r(x)\}$ a basis for $\mathbb{P}_2$? Justify your answer.

19. (4 points) Find the standard matrix for the combination of linear transformations $S \circ T$ if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformations which rotates vectors about the origin by $\frac{\pi}{3}$ radians counter-clockwise, and $S : \mathbb{R}^2 \to \mathbb{R}^2$ projects vectors onto the x-axis.

20. (6 points) Let $\mathcal{H} = \left\{ A \in M_{2\times 2} : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = A \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

(a) Given that $\mathcal{H}$ is a subspace, find a basis for it.

(b) What is the dimension of $\mathcal{H}$?

(c) Can $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$ be written as a linear combination of your basis vectors? Justify.

21. (6 points) Complete each of the following sentences with MUST, MIGHT, or CANNOT.

(a) Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be distinct nonzero vectors in $\mathbb{R}^3$. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ ____________ be equal to $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$.

(b) If A is a square matrix and $A^4 - 2A^2 + A = I$, then A ____________ be invertible.

(c) If A and B are $n \times n$ matrices such that $AB = B$, then A ____________ be an identity matrix.

(d) If $A\mathbf{x} = \mathbf{b}$ has two distinct solutions then the columns of A ____________ be linearly dependent.

(e) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lines in $\mathbb{R}^2$, where $\mathcal{L}_1$ does not pass through the origin and $\mathcal{L}_2$ passes through the origin. If there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathcal{L}_1) = \mathcal{L}_2$, then T ____________ be one-to-one.

(f) If A is an $m \times n$ matrix and $\text{Nul}(A) = \mathbb{R}^n$, then A ____________ be a $m \times n$ zero matrix.

 Question 21 continues on the next page.

ANSWERS

1. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$

2. $A^n\mathbf{x} = A^{n-1}A\mathbf{x} = A^{n-1}\mathbf{0} = \mathbf{0}$ 3. $p(x) = 10 + 9x + 5x^2$

4. $\left\{ \begin{bmatrix} 4 \\ 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \\ -1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 5 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -12 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \\ -1 \end{bmatrix} \right\}$ (for example)

5. (a) $A = (B^{-1} + 2I)^{-1}C$ or $A = B(I + 2B)^{-1}C$ (depending on approach) (b) $\begin{bmatrix} 3 & 1 \\ -7 & 0 \end{bmatrix}$

6. $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0 \Rightarrow \mathbf{u} \cdot \mathbf{u} + 0(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v} \cdot \mathbf{v} = 0 \Rightarrow ||\mathbf{u}||^2 - ||\mathbf{v}||^2 = 0 \Rightarrow ||\mathbf{u}|| = ||\mathbf{v}||$

7. (a)$\det(A) = 2, \det(B) = 5$ (b) $\frac{1}{2}$ (c) 405

8. 3 9. (a) $k = 0$ (b) $k \neq 0, 1$ (c) $k = 1$

10. $[A^T(4A)]^T = (4A)^T(A^T)^T = 4A^TA = A^T(4A)$

11. (a) $R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

12. (a) 5 (b) $\begin{bmatrix} 6 & -3 & 1 \end{bmatrix}$ (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/6 \\ 0 \\ 1 \end{bmatrix} \right\}$ (others possible)

13. $A = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & -2 & 4 \\ 0 & -3 & 2 & 4 \\ 0 & 0 & 14 & 28 \end{bmatrix}$

14. (a) $\begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 & -2 & 2 \\ 0 & 2 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & -3 \end{bmatrix}$

15. (a) 1 unit (b) $(\frac{5}{3}, \frac{1}{3}, \frac{7}{3})$ (c) $\frac{2\sqrt{5}}{5}$ (d) $\frac{3}{2}$ units^2

16. $-2x - 6y + z = -12$ 17. (a) No (b) Yes 18. (a) $r(x) = -6\,p(x) + 5\,q(x)$ (b) No

18. $\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 0 \end{bmatrix}$ 19. (a) $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$ (for example) (b) 2 (c) No

20. (a) CANNOT (b) MUST (c) MIGHT (d) MUST (e) CANNOT (f) MUST