

DDC Final Exam (May 2018)

1. [1] Write $14e^{i\frac{3\pi}{2}}$ in rectangular form.
2. [1] Find i^{-63} . Write your answer in rectangular form.
3. [2] Find the real and imaginary parts of $\frac{4}{5-3i}$.
4. [2] Sketch the set $\{z \in \mathbb{C} : |z-4-3i| = 3\}$ in the complex plane.
5. [6] Given $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\mathbf{x} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$ find: (a) $[\mathbf{x}]_{\mathcal{B}}$ (b) $[\mathbf{x}]_{\mathcal{E}}$ (c) \mathbf{y} (d) $[\mathbf{y}]_{\mathcal{E}}$
6. [5] Given that the eigenvalues of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ are $\lambda = 1, 2$, diagonalize A (find P and D s.t. $A = PDP^{-1}$).
7. [4] Define $T : \mathbb{P}_1 \rightarrow \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(2) \\ p(3) \end{bmatrix}$.
- (a) Find the matrix of T relative to the basis $\{1, t\}$ for \mathbb{P}_1 and the standard basis for \mathbb{R}^2 .
- (b) Find the matrix of T relative to the basis $\mathcal{B} = \{t, 1 - 2t\}$ for \mathbb{P}_1 and the basis $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$ for \mathbb{R}^2 .
8. [5] Let $A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$.
- (a) Find the eigenvalues of A .
- (b) Factor A into SR where S is a scaling and R is a rotation. Be sure to identify the angle of rotation.
9. [5] Let $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$ where O_k is the owl population and R_k is the wood rat population in a certain region at time k . Assume further that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where $A = \begin{bmatrix} 0.1 & 0.6 \\ -0.3 & 1 \end{bmatrix}$.
- (a) Find a formula for \mathbf{x}_k , given that $\mathbf{x}_0 = \begin{bmatrix} 110 \\ 70 \end{bmatrix}$.
- (b) Does the wood rat population grow or decline?
10. [5] Write y as the sum of a vector in $W = \text{span}\{u_1, u_2\}$ and a vector in W^\perp . Use the dot product.
- (a) $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
- (b) $y = \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$
11. [5] Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.
- (a) Find an orthogonal basis for $\text{Col}A$.
- (b) Find a QR factorization of A .
12. [6] Let \mathbb{P}_2 (polynomials of degree at most two) have the inner product $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.
- (a) Use the Gram-Schmidt process to find an orthogonal basis for $H = \text{span}\{1, t^2\}$.
- (b) Find an **orthonormal** basis for H^\perp .
13. [6] Find the second order Fourier approximation of $f(x) = 1 + x$ over the interval $[0, 2\pi]$.
14. [6] Consider the quadratic form on \mathbb{R}^2 : $Q(\mathbf{x}) = -6x_1x_2$.
- (a) Find a symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.
- (b) Make a change of variable $\mathbf{x} = P\mathbf{y}$, P orthogonal, that transforms Q into a quadratic form with no cross-product term. Clearly state P and the new quadratic form.
- (c) Classify Q .
15. [6] Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.
- Hint: The eigenvalues of $A^T A$ are 18 and 0.
16. [5] Find the minimum polynomial of $A = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$.
- Hint: the eigenvalues are 0 and -1 .
17. [6] Given $H = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$, find:
- (a) H_1 and H_2 , subspaces of \mathbb{R}^3 , such that $H \oplus H_1 \oplus H_2 = \mathbb{R}^3$
- (b) H_3 , a subspace of \mathbb{R}^3 , such that $H^\perp \oplus H_3 = \mathbb{R}^3$
- (c) H_4 , a subspace of \mathbb{R}^3 , such that $H \oplus H_4 = \left\{ \begin{bmatrix} 2x \\ x+y \\ 2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$
18. [5] Find a primary decomposition of $A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 6 & 0 & 0 \end{bmatrix}$. Hint: $m_A(\lambda) = \lambda^2(\lambda - 2)$.
19. [2] For which value(s) of $a \in \mathbb{R}$ is the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & 4 \end{bmatrix}$ diagonalizable?
20. [3] Let $V = \text{span}\{f_1, f_2\}$ where $f_1 = 4\sin x - \cos x$, $f_2 = -2\sin x + 2\cos x$, $g_1 = 2\sin x + \cos x$, $g_2 = 3\cos x$. Find the change of basis matrix from $\mathcal{B} = \{f_1, f_2\}$ to $\mathcal{C} = \{g_1, g_2\}$.
21. [4] Let $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ where $a, b \in \mathbb{R}$ and a and b are not both zero.
- (a) Find the eigenvalues of A .
- (b) Describe the eigenspaces of A .
- (c) Find an a and b that make A **not** diagonalizable.

22. [2] Let A be a 2×2 real symmetric matrix with two distinct eigenvalues. If $\begin{bmatrix} b \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ are eigenvectors of A , what are the possible values of b ?
23. [3] Let $A = \begin{bmatrix} 3/5 & c & 4/5 \\ a & 1 & e \\ b & d & f \end{bmatrix}$ be orthogonal where $a, b, c, d, e, f \in \mathbb{R}$ and $b > 0$. Find a, b, c, d, e , and f .

24. [2]

- (a) Find 2×2 matrices A and B , $A \neq B$, that have the same characteristic polynomial and the same minimum polynomial.
- (b) Find 2×2 matrices A and B that have the same characteristic polynomial but different minimum polynomials.

25. [3] Let $z, w \in \mathbb{C}$ such that $\bar{z}w \neq 1$ and either $|z| = 1$ or $|w| = 1$. Show that $\left| \frac{z-w}{1-\bar{z}w} \right| = 1$

Answers

1. $-14i$
2. i
3. $\frac{10}{17}, \frac{6}{17}$
4. Circle of radius 3 in the complex plane, centered at $4+3i$.
5. (a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 11 \\ -10 \end{bmatrix}$ (d) $\begin{bmatrix} 11 \\ -10 \end{bmatrix}$
6. $P = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
7. (a) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
 (b) $\begin{bmatrix} 1/2 & 0 \\ 0 & -1 \end{bmatrix}$
8. (a) $1 \pm i\sqrt{3}$
 (b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$, $\theta = -\pi/3$
9. (a) $\mathbf{x}_k = 30(0.7)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 40(0.4)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
 (b) $0 < \lambda < 1$ (both) \implies decline.
10. (a) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$
 (b) $\begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$
11. (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$(b) \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

12. (a) $\{1, t^2 - \frac{2}{3}\}$

(b) $\{\frac{1}{\sqrt{2}}t\}$

13. $1 + \pi - 2 \sin t - \sin 2t$

14. (a) $\begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

(b) $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $Q' = 3y_1^2 - 3y_2^2$

(c) indefinite

15. $\begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

16. $\lambda(\lambda+1)^2$

17. (a) $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) H

(c) $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

18. $A = PCP^{-1}$ where $P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} -3 & -3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

19. $a \neq 3, 4$

20. $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

21. (a) $\lambda = 0, a + b$

(b) $E_0 = \text{span}\left\{ \begin{bmatrix} -b \\ a \end{bmatrix} \right\}$, $E_{a+b} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $a = 1, b = -1$

22. $b = -2, \frac{9}{2}$

23. $a = c = d = e = 0, b = \frac{4}{5}, f = \frac{-3}{5}$

24. (a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

25.

$$\begin{aligned} \left| \frac{z-w}{1-\bar{z}w} \right|^2 &= \left(\frac{z-w}{1-\bar{z}w} \right) \overline{\left(\frac{z-w}{1-\bar{z}w} \right)} \\ &= \left(\frac{z-w}{1-\bar{z}w} \right) \left(\frac{\bar{z}-\bar{w}}{1-z\bar{w}} \right) \\ &= \frac{|z|^2 - z\bar{w} - \bar{z}w + |w|^2}{1 - z\bar{w} - \bar{z}w + |z|^2|w|^2} \end{aligned}$$

This last expression is 1 if $|z| = 1$ or $|w| = 1$.