1. (4 points) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 2 \\ 1 & 2 & a^2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$. Find the value(s) of a for which the equation $A\mathbf{x} = \mathbf{b}$ has:

(a) a unique solution.

(b) infinitely many solutions.

(c) no solution.

2. (4 points) Find the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $p(1) = -1, p(-1) = 1, p'(1) = -7$, and $p'(-1) = -3$.

3. (6 points) Given the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 4 & 0 & 8 & 4 & 4 \\ 2 & 0 & 3 & 2 & 1 \end{bmatrix}$, find a basis for each of the following subspaces.

(a) $\mathrm{Col}(A)$

(b) $\mathrm{Row}(A)$

(c) $\mathrm{Nul}(A^T)$

4. (2 points) Suppose $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Show that $\mathbf{w} = 3\mathbf{u} - 4\mathbf{v}$ is a solution to $A\mathbf{x} = 3\mathbf{b}$.

5. (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote the vertical expansion by a factor of 2 and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ denote the (counter-clockwise) rotation about the origin by $\pi/2$ radians.

(a) Find the standard matrix of the composite $S \circ T$.

(b) Let $\mathcal{R}$ denote the triangular region with vertices $(-1, -1), (4, 0)$, and $(3, 2)$. Make two sketches, one of $\mathcal{R}$, and the other of the image $(S \circ T)(\mathcal{R})$.

6. Let $A = \begin{bmatrix} B & 0 \\ C & 2I \end{bmatrix}$, where B is invertible.

(a) (3 points) Find an expression for the partitioned matrix A^{-1}.

(b) (3 points) Use your work from part (a) to find the inverse of the matrix $\begin{bmatrix} 2 & 4 & 0 & 0 & 0 \\ -1 & -5 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ 2 & -3 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{bmatrix}$.

(c) (1 point) Use your previous result to find the inverse of $\begin{bmatrix} 2 & -1 & -1 & 2 & 2 \\ 4 & -5 & 1 & -3 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

7. Consider the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 2 \\ 1 & 3 & 12 \end{bmatrix}$.

(a) (3 points) Find A^{-1} by row reduction.

(b) (4 points) Express the matrix A as a product of elementary matrices.

8. (3 points) Let A and B be invertible $n \times n$ matrices. Given that B is symmetric, determine whether the matrix $AB^{-1}A^T - B$ is also symmetric. Justify your answer.

9. (3 points) Solve for the matrix X in the equation below:

$$(3XB)^{-1} + A = X^{-1}$$

Assume that all matrices involved are invertible.

10. (4 points) Given the matrix $A = \begin{bmatrix} 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -3 \end{bmatrix}$, find the determinant of A.

11. (6 points) Given A, B, and C are 2×2 matrices such that $\det(A) = 3$, $\det(B) = -2$, and $\det(C) = 0$, evaluate the following determinants.

(a) $\det((3AB^2)^{-1})$

(b) $\det(AC + BC)$

(c) $\det(A^{-1} + \text{adj}(A))$

12. (7 points) Let $\mathcal{H} = \{A \in M_{2\times 2} \, : \, \det A = 0\}$.

(a) Find two matrices in $\mathcal{H}$, neither of which is a scalar multiple of the other.

(b) Is $\mathcal{H}$ closed under addition?

(c) Is $\mathcal{H}$ closed under scalar multiplication?

(d) Is $\mathcal{H}$ a subspace of $M_{2\times 2}$?

13. (4 points) Let $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\mathcal{S} = \{A \in \mathbb{M}_{2\times 2} \, : \, A\mathbf{v} = \mathbf{0}\}$. Find a basis for $\mathcal{S}$.

14. (4 points) Let W be the set of all polynomials $\mathbf{p}$ in $\mathbb{P}_3$ such that $\mathbf{p}(1) = 0$ and $\mathbf{p}'(-1) = 0$. Find a basis for W.

15. (6 points) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find an equation of the form $ax_1 + bx_2 + cx_3 = d$ for the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

(b) Show that the line $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + t \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$ is entirely contained on the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

16. Consider the plane $\mathcal{P}:\ 2x - 4y + 2z = 6$, the line $\mathcal{L} : \mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and the point $Q(1, 6, 0)$.

(a) (1 point) Find an equation for the line through the origin and the point Q.

(b) (2 points) Find the cosine of the angle between the plane $\mathcal{P}$ and the $yz-$plane.

(c) (3 points) Find the distance from the point Q to the line $\mathcal{L}$.

(d) (4 points) Find the point on the plane $\mathcal{P}$ that is closest to the point Q.

(e) (3 points) Find an equation of the form $ax + by + cz = d$ for the plane that contains the line $\mathcal{L}$ and is perpendicular to the plane $\mathcal{P}$.

17. (4 points) Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ k \\ k^2 \end{bmatrix}$.

(a) Find all values of k for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) Find a unit vector that is orthogonal to $\mathbf{u}$.

18. (3 points) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a set of linearly independent vectors in $\mathbb{R}^3$.

(a) Simplify $\mathbf{u} \cdot [(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})]$.

(b) True or false: the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ has the same volume as the parallelepiped with sides $\mathbf{u}$, $\mathbf{v} - \mathbf{u}$, and $\mathbf{w} - \mathbf{u}$.

19. (3 points) Show that if $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent, then $\{\mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}\}$ is also linearly independent.

20. (4 points) Complete the following statements with "must", "might", or "cannot", as appropriate.

(a) If A is a product of elementary matrices, then $\det(A)$ __________ equal zero.

(b) Two lines in $\mathbb{R}^3$ that are both perpendicular to a third line __________ be parallel.

(c) If matrix AB is invertible, then A __________ be invertible.

(d) Given a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$, the kernel of T __________ be a plane.

ANSWERS

1. (a) $a \neq -1, a \neq 1$, and $a \neq 4$ (b) $a = 4$ (c) $a = -1$ or $a = 1$

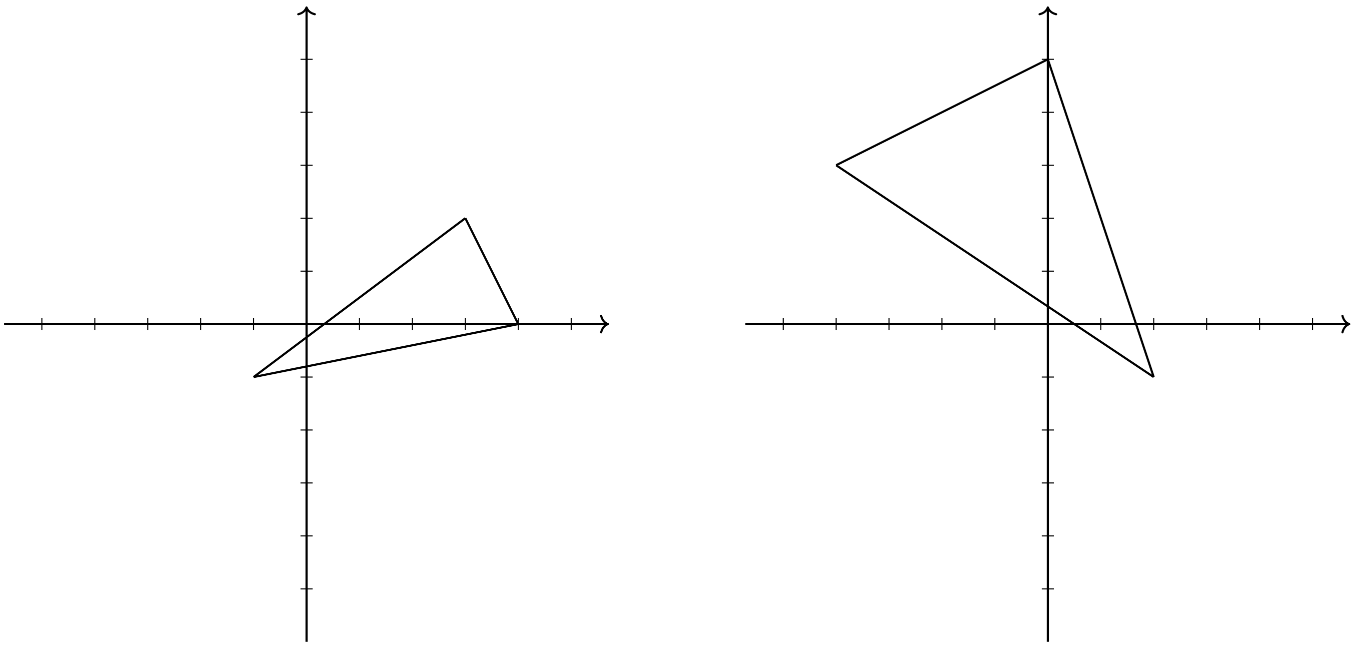
2. $p(x) = x^4 - 2x^3 - 3x^2 + x + 2$

3. (a) $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix} \right\}$ (b) $\{[1\ 0\ 2\ 1\ 1], [0\ 0\ 1\ 0\ 1]\}$ (c) $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \right\}$

4. $A\mathbf{w} = A(3\mathbf{u} - 4\mathbf{x}) = 3A\mathbf{u} - 4A\mathbf{v} = 3\mathbf{b} - 4 \cdot \mathbf{0} = 3\mathbf{b}$

5. (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

(b)



6. (a) $\begin{bmatrix} B^{-1} & 0 \\ -\frac{1}{2}CB^{-1} & \frac{1}{2}I \end{bmatrix}$ (b) $\begin{bmatrix} 5/6 & 2/3 & 0 & 0 & 0 \\ -1/6 & -1/3 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 0 & 0 \\ -13/12 & -7/6 & 0 & 1/2 & 0 \\ -3/4 & -1/2 & 0 & 0 & 1/2 \end{bmatrix}$ (c) Transpose of answer in part (b)

7. (a) $A^{-1} = \begin{bmatrix} -24 & 6 & 1 \\ 4 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 24 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

8. Yes, as $(AB^{-1}A^T - B)^T = (AB^{-1}A^T)^T - B^T = A(B^{-1})^T A^T - B = A(B^T)^{-1}A^T - B = AB^{-1}A^T - B$

9. $X = A^{-1}(I - \frac{1}{3}B^{-1})$

10. $\det(A) = 3$ 11. (a) $\dfrac{1}{108}$ (b) 0 (c) $\dfrac{16}{3}$

12. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (many answers)

(b) No. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}$ however $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \mathcal{H}$

(c) Yes. Given $A \in \mathcal{H}, k \in \mathbb{R}$, $\det(kA) = k^2 \det A = k^2 \cdot 0 = 0$

(d) No, as $\mathcal{H}$ is not closed under vector addition.

13. $\left\{ \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} \right\}$ 14. $\{3x^3 + 2x^2 - 5x, 2x^3 + 3x^2 - 5\}$

15. (a) $2x_1 + x_2 - 5x_3 = 0$ (b) $2x_1 + x_2 - 5x_3 = 2(2+9t) + (6+2t) - 5(2+4t) = 0$

16. (a) $\mathbf{x} = t \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$ (b) Using $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\cos(\theta) = \dfrac{1}{\sqrt{6}}$ (c) $\sqrt{3}$ units

(d) $\left(\dfrac{10}{3}, \dfrac{4}{3}, \dfrac{7}{3} \right)$ (e) $y + 2z = 7$

17. (a) $k = -2$ and $k = 3$ (b) $\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (many answers)

18. (a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ (b) True

19. Show that if $c_1(\mathbf{a} - \mathbf{b}) + c_2(\mathbf{a} + \mathbf{b}) = 0$, then $c_1 = 0$ and $c_2 = 0$.

$c_1(\mathbf{a} - \mathbf{b}) + c_2(\mathbf{a} + \mathbf{b}) = 0 \rightarrow c_1\mathbf{a} - c_1\mathbf{b} + c_2\mathbf{a} + c_2\mathbf{b} = 0 \rightarrow (c_1 + c_2)\mathbf{a} + (-c_1 + c_2)\mathbf{b} = 0$

As $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, $c_1 + c_2 = 0$ and $-c_1 + c_2 = 0$. Solving this system of two equations gives $c_1 = 0$ and $c_2 = 0$.

20. (a) cannot (b) might (c) might (d) might