Solutions

(Marks)

(2) 1. Evaluate

(a) $\frac{700!}{697!} = 700\cdot699\cdot698$

$= 341\,531\,400$

(b) $_{700}C_{697} = \frac{700!}{(700-697)!\,697!} = \frac{700\cdot699\cdot698}{3\cdot2\cdot1}$

$= 56\,921\,900$

(6) 2. How many four letter words (with or without meaning) can be made using the letters: $\{t, u, r, i, n, g\}$ if...

(a) repetition is allowed?

permutation $6^4 = 1296$

(b) repetition is not allowed?

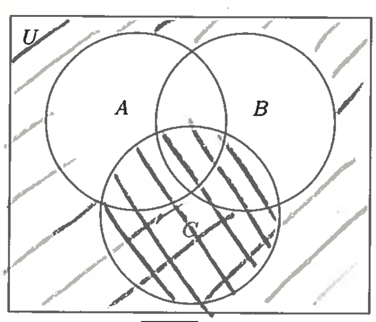
permutation $_6P_4 = \frac{6!}{(6-4)!} = \frac{6\cdot5\cdot4\cdot3}{2\cdot1} = 360$

(3) 3. A child gets to fill a bag with 20 jellybeans, which come in four colours: red, yellow, blue, and green. How many ways can the child choose the jellybeans?

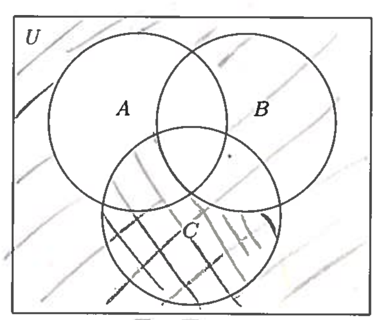
Combination with repetition # of elements $4+20-1$

$_{23}C_{20} = \frac{23!}{20!\,3!} = \frac{23\cdot\overset{11}{\cancel{22}}\cdot\overset{7}{\cancel{21}}}{\cancel{3}\cdot\cancel{2}\cdot1} = 1771$

(5) 4. Use provided Venn diagrams to verify the identity $\overline{A\cup B}\cap C = \overline{A}\cap(\overline{B}\cap C)$. State clearly whether the result in each case is cross-hatched or anything hatched.



$\overline{A\cup B}\cap C$



$\overline{A}\cap(\overline{B}\cap C)$

cross-hatched the same cross-hatched

(Marks)

(6) 5. Let $U = \{0,1,2,3,4,5,6,7,8,9\}$ be a universal set. Let $E = \{0,2,4,6,8\}$, $T = \{0,3,6,9\}$, $F = \{0,5\}$.

(a) Find $E \cap T$.

$= \{0,6\}$

(b) Find $\overline{E} \cap F$

$= \{5\}$

(c) Find $\overline{E \cup T \cup F}$

$= \{1,7\}$

(d) How many proper subsets does U have?

$2^{10} - 1 = 1023$

(e) How many subsets of E have exactly three elements?

E - 5 elements $\quad {}_5C_3 = \frac{5!}{2!\,3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$

(f) How many subsets of T have at least three elements?

3 or 4

${}_4C_3 + {}_4C_4 = 4 + 1 = 5$

(4) 6. Use truth tables to determine if the logical statements are equivalent: $(p \underline{\vee} q) \underline{\vee} r$ and $p \underline{\vee} (q \underline{\vee} r)$.

p	q	r	$(p \underline{\vee} q)$	$(p \underline{\vee} q) \underline{\vee} r$	$(q \underline{\vee} r)$	$p \underline{\vee} (q \underline{\vee} r)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

the same so equivalent statements

(Marks)

(3) 7. Use truth tables to determine if the statement is a tautology, contradiction, or neither: $(p\overline{\vee}q) \rightarrow (p\overline{\wedge}q)$.

p	q	$p\bar{\vee}q$	$p\bar{\wedge}q$	$(p\bar{\vee}q) \rightarrow (p\bar{\wedge}q)$
T	T	F	F	T
T	F	F	T	T
F	T	F	T	T
F	F	T	T	T

tautology

$p \rightarrow q$

(3) 8. Consider the statement: "If the shoe fits, then you wear it."
Label each of the following as its inverse, converse, or contrapositive:

(a) If you wear it, then the shoe fits.

$q \rightarrow p$ converse

(b) If you don't wear it, then the shoe doesn't fit.

$\sim q \rightarrow \sim p$ contrapositive

(c) You don't wear it if the shoe doesn't fit.

$\sim p \rightarrow \sim q$ inverse

(2) 9. State one of the **Idempotent Properties** for Logic.

$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$

(2) 10. State one of the **Complement Properties** for Set Theory.

$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$

(2) 11. State one of the **Associative Properties** for Boolean Algebra.

$A + (B + C) = (A + B) + C$ $A(BC) = (AB)C$

(Marks)

(4) 12. Translate the following argument into symbolic logic and then use truth tables to determine if it is valid:

H: If tax rates are lowered, then business would thrive.
If business thrives, then more tax would be paid to the government.

C: If tax rates are lowered, then more tax would be paid to the government.

p : Tax rates are lowered q : Business thrives r : more tax

H: $p \to q$
$q \to r$

C: $p \to r$

p	q	r	$p \to q$	$q \to r$	H	C	$H \to C$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

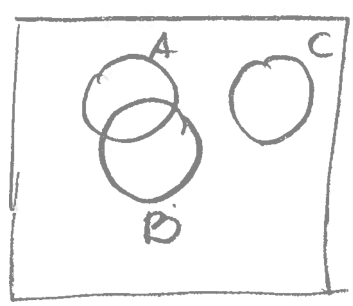
Valid

(3) 13. Use Venn Diagrams to determine if the following argument is valid. Be sure to name and label your sets.

H: No penguins fly.
Some birds are penguins.

C: Some birds fly.

A = penguins
B = Birds
C = Fly



invalid

A

(Marks)

(3) 14. Create a Boolean Table for the following expression: $\overline{A \cdot B \cdot C}(A \cdot C + B)$.

A	B	C	$\overline{ABC}$	$A\cdot C$	$AC+B$	$\overline{ABC}\,(AC+B)$
1	1	1	0	1	1	0
1	1	0	1	0	1	1
1	0	1	1	1	1	1
1	0	0	1	0	0	0
0	1	1	1	0	1	1
0	1	0	1	0	1	1
0	0	1	1	0	0	0
0	0	0	1	0	0	0

(6) 15. Simplify each boolean expression. State the properties that you are using at each step.

(a) $A\overline{B} + BAC$

$A\overline{B} + ABC$ commut.

$A(\overline{B} + BC)$ dist

$A(\overline{B}+B)(\overline{B}+C)$ dist

$A(1)(\overline{B}+C)$ complement

$A(\overline{B}+C)$ identity

(b) $(A + \overline{B+C})(B\overline{C} + A)$

$(A + \overline{B}\circ\overline{C})(B\overline{C}+A)$ de Morgan

$(A+\overline{B})(A+\overline{C})(B+A)(\overline{C}+A)$ dist

idempotent

$(A+\overline{B})(B+A)(A+\overline{C})$

$(A+\overline{B}B)(A+\overline{C})$ distributive

$(A+0)(A+\overline{C})$ complement identity

$A(A+\overline{C})$ idemp.

$AA + A\overline{C}$ identity

~~AA+AC~~

$A\cdot 1 + A\cdot\overline{C}$

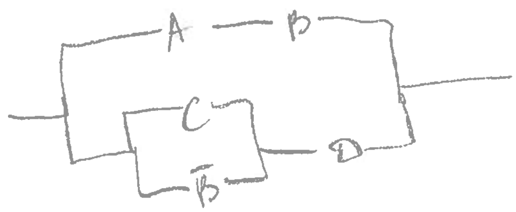
$A(1+\overline{C})$ dist.

$A(1)$ prop 1

A identity

(Marks)

(2) 16. Draw a network diagram that represents the following Boolean Expression: $AB + (C + \overline{B})\ D$



(5) 17. (a) Solve $\begin{aligned} 3y - 6x &= 9 \\ x - y &= 1 \end{aligned}$ by substitution or elimination.

(b) Graph both lines to confirm your result.

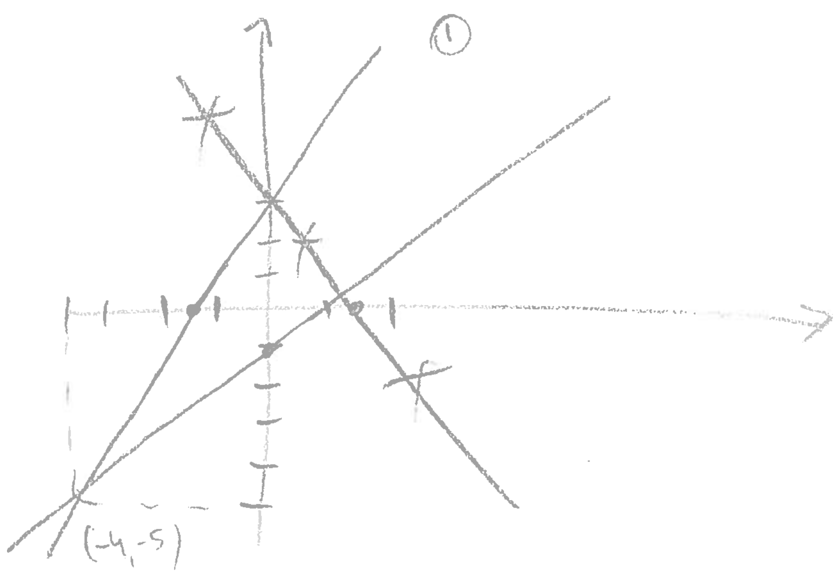
$x = y + 1$

$3y - 6(y+1) = 9$

$-3y - 6 = 9$

$-3y = 15$

$y = -5$ $\quad x = -5 + 1 = 4$ $\quad$ a) $(-4, -5)$



1

x	y
0	3
$-\frac{3}{2}$	0

2

x	y
0	-1
1	0

(Marks)

(10) 18. Given: $A = \begin{bmatrix} -2 & 3 & -1 \\ 5 & 4 & 0 \\ -1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 & -4 & 1 \\ 3 & -1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

find each of the following, if possible. If an operation is not possible, say why.

(a) BA (b) BI_2 (c) $D^2 - BC$ (d) $B^T - 2C$

a) $$BA = \begin{bmatrix} 4-20-1 & -6-16+2 & 2-0+4 \\ -6-5-4 & 9-4+8 & -3-0+16 \end{bmatrix} = \begin{bmatrix} -17 & -20 & 6 \\ -15 & 13 & 13 \end{bmatrix}$$

b) undefined $(2\times \underbrace{3)\cdot(2}_{\neq} \times 2)$

c) $$D^2 - BC = D \cdot D - BC$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} -2 & -4 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -21 \\ -7 & 28 \end{bmatrix} - \begin{bmatrix} 13 & -18 \\ 11 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ -18 & 29 \end{bmatrix}$$

d) $$\begin{bmatrix} -2 & 3 \\ -4 & -1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ -4 & 8 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -9 \\ -5 & 4 \end{bmatrix}$$

(Marks)

(3) 19. Given that the augmented matrix of the system $AX = B$, reduces to the form below, give all solutions (if any)of the system. If there are infinitely many solutions express each variable in terms of free variable(s).

(a) $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right]$

$x = 1 + 3z$
$y = -2$
$z = z$ (any number)

No solution

(9) 20. Use matrices and row reduction to solve, if possible, the following systems. If there are infinitely many solutions express each variable in terms of free variable(s).

(a) $\begin{array}{rcrcrcl} 3x & - & 7y & + & 4z & = & 10 \\ -x & - & 2y & + & 3z & = & 1 \\ x & + & y & + & 2z & = & 8 \end{array}$ (b) $\begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 4 \\ 3x & - & y & + & 5z & = & 2 \\ 4x & + & y & + & 2z & = & 6 \end{array}$

$\left[\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \end{array}\right]$ $R_1 \Leftrightarrow R_3$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$ $R_2 + R_1$, $R_3 - 3R_1$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$ $-R_2$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$ $R_3 + 10R_2$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$ $-\frac{1}{52}R_3$

$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $R_1 - 2R_3$, $R_2 + 5R_3$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $R_1 - R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & 2 & 6 \end{array}\right]$ $R_2 - 3R_1$, $R_3 - 4R_1$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & 14 & -10 \end{array}\right]$ $R_3 - R_2$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $-\frac{1}{7}R_2$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 - 2R_2$

$4 - \frac{20}{7} = \frac{8}{7}$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x = \frac{8}{7} - z$

$y = \frac{10}{7} + 2z$

$z = z$

(Marks)

(5) 21. Use row operations to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] R_3 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] -R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_2 + 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] R_1 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -40+26+15 & 16-10-6 & 9-6 \\ -80+65+15 & 32-25-6 & 18-15 \\ -40+40 & 16-16 & 9-8 \end{bmatrix}$$

$$= I$$

(3) 22. Given $A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 6 & 4 \\ -1 & 2 & 5 \end{bmatrix}$ show that A^{-1} does not exist.

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -1 & 1 & 0 & 0 \\ 1 & 6 & 4 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] R_1 + R_3 \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 1 \\ 1 & 6 & 4 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

inverse doesn't exist

(Marks)

(5) 23. Given the linear system $\begin{array}{rcrcr} 3x & + & 2y & = & -5 \\ x & + & 6y & = & 1 \end{array}$

(a) Write this system as a matrix equation $AX = B$.

$$\begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

(b) Find the inverse matrix of A.

$$A^{-1} = \frac{1}{18-2}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

(c) Use A^{-1} to solve the system.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-30-2}{16} \\ \frac{5+3}{16} \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{1}{2} \end{bmatrix}$$

(4) 24. Use mathematical induction to prove that the following statement P_n is true for all positive integers n:

$$P_n: \ 5^1 + 5^2 + 5^3 + \cdots + 5^n = \frac{5}{4}(5^n - 1)$$

P_1: $5^1 = \frac{5}{4}(5^1 - 1)$

$5 = \frac{5}{4} \cdot 4$ true

P_k: $5 + 5^2 + \ldots + 5^k = \frac{5}{4}(5^k - 1)$

P_{k+1}: $\underbrace{5 + 5^2 + \ldots + 5^k}_{} + 5^{k+1} = \frac{5}{4}(5^{k+1} - 1)$

$\frac{5}{4}(5^k - 1) + 5^{k+1}$

$\frac{5^{k+1}}{4} - \frac{5}{4} + 5^{k+1}$

$\frac{5 \cdot 5^{k+1}}{4} - \frac{5}{4}$

$\frac{5}{4}(5^{k+1} - 1)$

$\Rightarrow$ P_n true for all $n \geq 1$