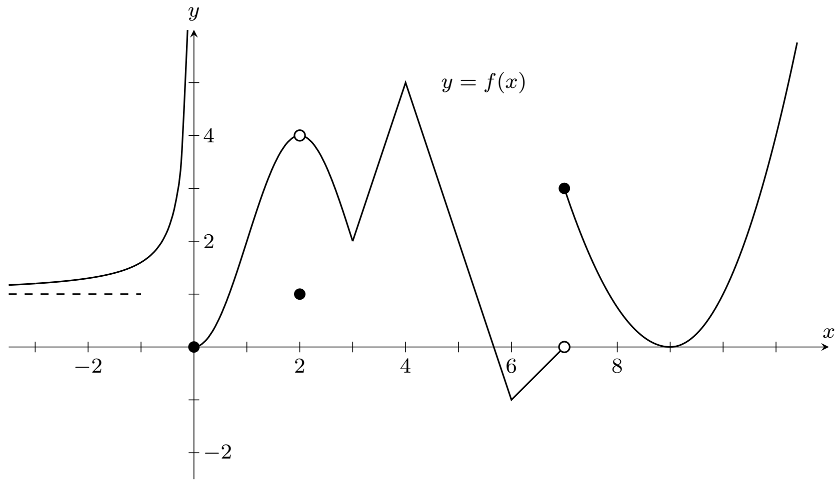
1. (5 points) Given the graph of f below, evaluate each of the following. Use ∞, $-\infty$ or "does not exist" where appropriate.



(a) $\lim\limits_{x\to-\infty} f(x)$

(b) $\lim\limits_{x\to 0^-} [5 - f(x)]$

(c) $\lim\limits_{x\to 7^-} \dfrac{1}{f(x)}$

(d) $\lim\limits_{h\to 0} \dfrac{f(5+h)-f(5)}{h}$

(e) List all values of x for which f is not differentiable.

2. (10 points) Evaluate the following limits. Where appropriate, indicate ∞ or $-\infty$ or explain why the limit does not exist.

(a) $\lim\limits_{x\to 5} \dfrac{2\sqrt{x-1}-\sqrt{x^2-9}}{2x^2-9x-5}$.

(b) $\lim\limits_{x\to 3} \dfrac{9-x^2}{x^2-|6-5x|}$.

(c) $\lim\limits_{x\to 0} \dfrac{\tan(\pi x)}{x}$.

(d) $\lim\limits_{x\to 0^+} \sqrt{\sqrt{x}+x^2}\cos\left(\dfrac{\pi}{x}\right)$.

(e) $\lim\limits_{x\to 0} \left(\dfrac{x-\cos x}{x^2}\right)$.

3. (3 points) List all horizontal asymptotes of the graph of $y = \dfrac{1+e^x}{e^x-5}$, or indicate that none exist, as appropriate.

4. (3 points) Consider the following function.

$$f(x) = \begin{cases} ax^2 - 5 & \text{if } x < 2, \\ a^2 & \text{if } x = 2, \\ x^2 + ax - 7 & \text{if } x > 2. \end{cases}$$

Is there a value of a that makes f continuous on $\mathbb{R}$? Fully support your answer.

5. (5 points) (a) State the limit definition of the derivative.

(b) Find $f'(x)$ using the limit definition of the derivative if $f(x) = \dfrac{1}{2x^2+5}$.

6. (15 points) Find $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y = \sqrt[3]{x^5} - \dfrac{3^x}{\ln(5x)} + 6e^{7\pi} - \sec\left(\dfrac{8}{x}\right)$

(b) $\sin(x^2y) = ye^x$

(c) $y = \dfrac{\tan^2(x)}{(8x^2-7)\sqrt{5x+1}}$ Use logarithmic differentiation.

(d) $y = [e^x(x^2-5)+\ln(x)]^9$

(e) $y = \left[\dfrac{\csc(x)}{3}\right]^{\log_2(x)}$

7. (3 points) Find an expression for $f^{(57)}(x)$ given that $f(x) = xe^x + 10x^{32}$.

8. (3 points) Find the points (x and y coordinates), if any, at which $f(x) = \ln(x^3 - 3x^2 - 9x + 10)$ has a horizontal tangent line.

9. (11 points) For the function

$$f(x) = x^4 + 4x^3$$

do the following.

(a) Find $f'(x)$ and $f''(x)$. Give your answers in factored form.

(b) Find the y-intercept and all x-intercepts.

(c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(d) Find the intervals of increase and decrease.

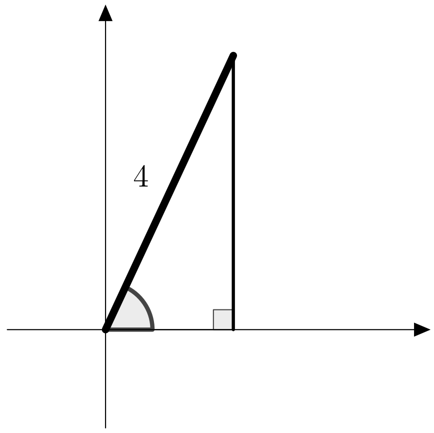
(e) Find the coordinates of all local maxima and minima.

(f) Find (1) the interval(s) where the graph is concave up and (2) the interval(s) where the graph is concave down.

(g) Find the coordinates of each inflection point.

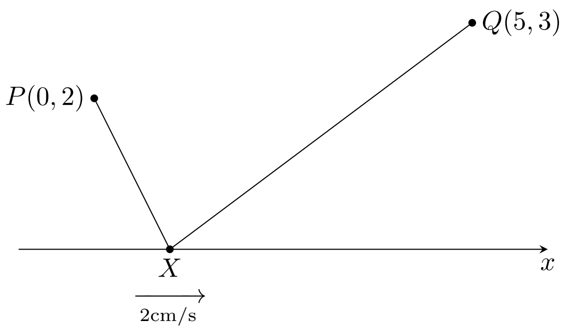
(h) Sketch the graph, making sure that your graph illustrates all these features.

10. (5 points) A line segment of length 4 rotates counter-clockwise about the origin from the positive x-axis to the positive y-axis. A triangle is formed by connecting the tip of the line segment to the x-axis perpendicularly (as in the image). What is the maximum area of the triangle?



11. (5 points) Find the largest and smallest values of the function $f(x) = (2x)^{2/3}(-x + 10) + 1$ on the interval $[-4, \frac{1}{2}]$.

12. (5 points) In the figure below, X is moving along the x-axis towards the right at a rate of 2cm/s. Find the rate at which $|PX| + |QX|$, the sum of the distances between X and the points $P(0, 2)$ and $Q(5, 3)$, is changing as X passes the point $(3, 0)$.



13. (3 points) Use the Mean Value Theorem to show that $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ for all $x > 0$. (If it helps, you may use the fact that this is equivalent to proving that $\sqrt[3]{1+x} - (1 + \frac{1}{3}x) < 0$ for all $x > 0$.)

14. (3 points) The position of an object along the x-axis is given by $x(t) = \sin^2(t) - \sin(t)$ for $t \geqslant 0$.

 Question 14 continues on the next page.

(a) Find the velocity function.

(b) Find the distance travelled by the object while $\pi \leqslant t \leqslant 2\pi$.

15. (12 points) Evaluate the following integrals.

(a) $\displaystyle\int \left[\frac{2}{3x^2} - \frac{4}{x} + 6^x - \pi^2\right] dx$

(b) $\displaystyle\int \frac{(2-\sqrt{x})^2}{2\sqrt{x}} dx$

(c) $\displaystyle\int_{\pi/3}^{\pi/4} \frac{\tan x}{\sec x} dx$

(d) $\displaystyle\int_{-4}^{4} \left(|x| - \sqrt{16 - x^2}\right) dx$ (Interpret the definite integral as areas.)

16. (3 points) Given $f''(x) = \dfrac{4}{x^{2/3}} - \dfrac{3}{x^{1/2}}$, $f'(1) = 12$ and $f(1) = 15$, find $f(x)$.

17. (4 points) Compute $\displaystyle\int_0^2 \left(4x - 3x^2\right) dx$ as a limit of Riemann Sums. You may find the following identities useful :

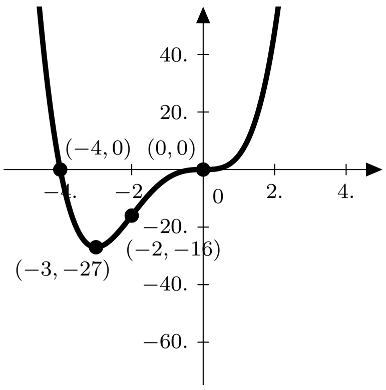
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad , \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

18. (2 points) Use the Fundamental Theorem of Calculus to find the derivative $f'(x)$ if $f(x) = \displaystyle\int_1^{1/x} \frac{1}{e^{1/t}+1} dt$. Simplify your answer.

Question 18 continues on the next page.

Answers

1. (a) 1 (b) $-\infty$ (c) $-\infty$ (d) -3 (e) $0, 2, 3, 4, 6, 7$
2. (a) $\frac{-3}{44}$ (b) -6 (c) π (d) 0 (e) $-\infty$
3. $y = 1$ on the right, and $y = \frac{-1}{5}$ on the left
4. No value of a can make $f(x)$ continuous on $\mathbb{R}$.
5. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ (b) $\lim\limits_{h\to 0} \dfrac{\frac{1}{2(x+h)^2+5} - \frac{1}{2x^2+5}}{h} = \dfrac{-4x}{(2x^2+5)^2}$
6. (a) $y' = \dfrac{5\sqrt[3]{x^2}}{3} - \dfrac{3^x(\ln(3)\ln(5x) - \frac{1}{x})}{[\ln(5x)]^2} + \dfrac{8\sec(\frac{8}{x})\tan(\frac{8}{x})}{x^2}$ (b) $y' = \dfrac{ye^x - 2xy\cos(x^2y)}{x^2\cos(x^2y) - e^x}$
(c) $y' = \dfrac{\tan^2(x)}{(8x^2-7)\sqrt{5x+1}}\left[\dfrac{2\sec^2(x)}{\tan(x)} - \dfrac{16x}{8x^2-7} - \dfrac{5}{2(5x+1)}\right]$ (d) $y' = 9[e^x(x^2-5)+\ln(x)]^8\left(e^x(x^2+2x-5) + \dfrac{1}{x}\right)$
(e) $y' = \left[\dfrac{\csc(x)}{3}\right]^{\log_2(x)}\left(\dfrac{\ln(\csc(x)) - \ln(3)}{x\ln(2)} - \log_2(x)\cot(x)\right)$
7. $f^{(57)}(x) = xe^x + 57e^x$
8. $x = -1$ (The function is undefined at $x = 3$.)
9. (a) $f'(x) = 4x^2(x+3)$, $f''(x) = 12x(x+2)$ (b) x-intercepts: $x = -4, 0$, y-intercept: $y = 0$ (c) $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$ (d) increasing on $[-3, \infty)$, decreasing on $(-\infty, -3]$ (e) local min: $(-3, -27)$, no local max (f) concave up on $(-\infty, -2]$ and $[0, \infty)$, concave down on $[-2, 0]$ (g) $(-2, -16)$ and $(0, 0)$ (h) See image.



10. 4 units^2
11. absolute max: 57, absolute min: 1
12. $\frac{2\sqrt{13}}{13}$ cm/s
13. **OPTION A:** If $f(x) = (1+x)^{1/3}$ then f is differentiable (and thus continuous) on $(-1, \infty)$. For $x > 0$, the mean value theorem applied to f on $[0, x]$ yields a real number x_0 such that $0 < x_0 < x$ and $f(x) = f(0) + f'(x_0)(x-0) = 1 + \frac{1}{3}x(1+x_0)^{-2/3} < 1 + \frac{1}{3}x$, since $(1+x_0)^{-2/3} < 1$.

OR **OPTION B:** Let $f(x) = \sqrt[3]{1+x} - (1 + \frac{1}{3}x)$, so $f'(x) = \frac{1}{3\sqrt[3]{(1+x)^2}} - \frac{1}{3}$. Note that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. The MVT states that there must exist a point $x_0 \in (0, a)$ such that $f'(x_0) = \frac{f(a)-f(0)}{a-0} = \frac{f(a)}{a}$ for any interval $(0, a)$, which implies that $a \cdot f'(x_0) = f(a)$ must hold for any $a > 0$. However, $f'(x)$ is always negative for $x > 0$, so $a \cdot f'(x_0)$ (and therefore $f(a)$) must always be less than zero when $a > 0$, i.e. $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ for all $x > 0$.
14. (a) $v(t) = 2\sin(t)\cos(t) - \cos(t)$ (b) 4 units
15. (a) $\frac{-2}{3x} - 4\ln|x| + \frac{6^x}{\ln(6)} - \pi^2 x + C$ (b) $4\sqrt{x} - 2x + \frac{\sqrt{x^3}}{3} + C$ (c) $\frac{1-\sqrt{2}}{2}$ (d) $16 - 8\pi$
16. $f(x) = 9\sqrt[3]{x^4} - 4\sqrt{x^3} + 6x + 4$
17. $\lim\limits_{n\to\infty}\left(\sum_{i=1}^{n} \dfrac{2}{n}\left[4\left(\dfrac{2}{n}i\right) - 3\left(\dfrac{2}{n}i\right)^2\right]\right) = 0$
18. $f'(x) = \frac{-1}{x^2(e^x+1)}$