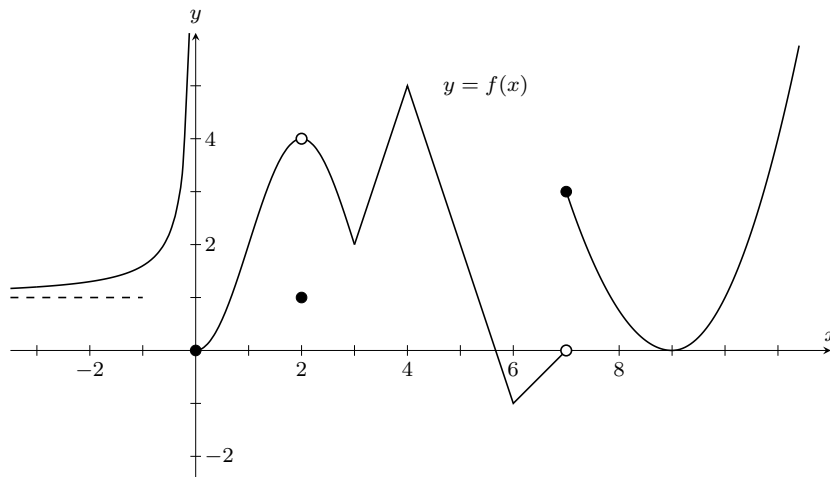


1. (5 points) Given the graph of f below, evaluate each of the following. Use ∞ , $-\infty$ or “does not exist” where appropriate.



- (a) $\lim_{x \rightarrow -\infty} f(x)$
- (b) $\lim_{x \rightarrow 0^-} [5 - f(x)]$
- (c) $\lim_{x \rightarrow 7^-} \frac{1}{f(x)}$
- (d) $\lim_{h \rightarrow 0} \frac{f(5+h) - f(5)}{h}$
- (e) List all values of x for which f is not differentiable.
2. (10 points) Evaluate the following limits. Where appropriate, indicate ∞ or $-\infty$ or explain why the limit does not exist.

(a) $\lim_{x \rightarrow 5} \frac{2\sqrt{x-1} - \sqrt{x^2-9}}{2x^2 - 9x - 5}$.

(b) $\lim_{x \rightarrow 3} \frac{9 - x^2}{x^2 - |6 - 5x|}$.

(c) $\lim_{x \rightarrow 0} \frac{\tan(\pi x)}{x}$.

(d) $\lim_{x \rightarrow 0^+} \sqrt{\sqrt{x} + x^2} \cos\left(\frac{\pi}{x}\right)$.

(e) $\lim_{x \rightarrow 0} \left(\frac{x - \cos x}{x^2} \right)$.

3. (3 points) List all horizontal asymptotes of the graph of $y = \frac{1 + e^x}{e^x - 5}$, or indicate that none exist, as appropriate.

4. (3 points) Consider the following function.

$$f(x) = \begin{cases} ax^2 - 5 & \text{if } x < 2, \\ a^2 & \text{if } x = 2, \\ x^2 + ax - 7 & \text{if } x > 2. \end{cases}$$

Is there a value of a that makes f continuous on \mathbb{R} ? Fully support your answer.

5. (5 points) (a) State the limit definition of the derivative.

(b) Find $f'(x)$ using the limit definition of the derivative if $f(x) = \frac{1}{2x^2 + 5}$.

6. (15 points) Find $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y = \sqrt[3]{x^5} - \frac{3^x}{\ln(5x)} + 6e^{7\pi} - \sec\left(\frac{8}{x}\right)$

(b) $\sin(x^2y) = ye^x$

(c) $y = \frac{\tan^2(x)}{(8x^2 - 7)\sqrt{5x + 1}}$ Use logarithmic differentiation.

(d) $y = [e^x(x^2 - 5) + \ln(x)]^9$

(e) $y = \left[\frac{\csc(x)}{3} \right]^{\log_2(x)}$

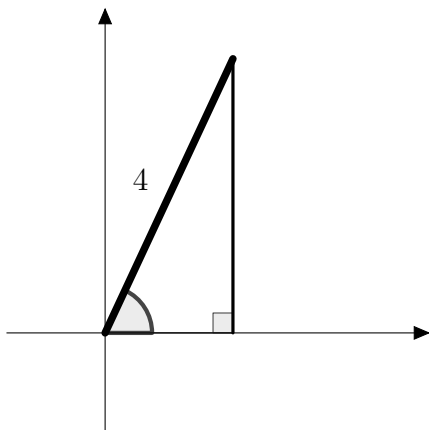
7. (3 points) Find an expression for $f^{(57)}(x)$ given that $f(x) = xe^x + 10x^{32}$.
8. (3 points) Find the points (x and y coordinates), if any, at which $f(x) = \ln(x^3 - 3x^2 - 9x + 10)$ has a horizontal tangent line.

9. (11 points) For the function

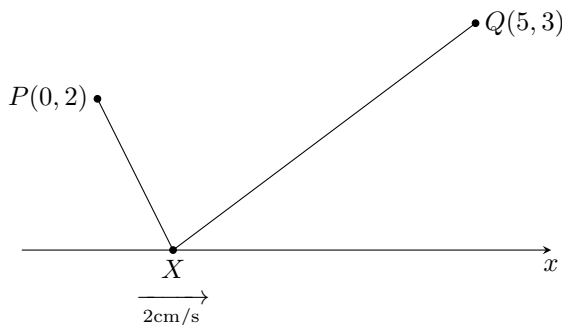
$$f(x) = x^4 + 4x^3$$

do the following.

- Find $f'(x)$ and $f''(x)$. Give your answers in factored form.
 - Find the y -intercept and all x -intercepts.
 - Find $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.
 - Find the intervals of increase and decrease.
 - Find the coordinates of all local maxima and minima.
 - Find (1) the interval(s) where the graph is concave up and (2) the interval(s) where the graph is concave down.
 - Find the coordinates of each inflection point.
 - Sketch the graph, making sure that your graph illustrates all these features.
10. (5 points) A line segment of length 4 rotates counter-clockwise about the origin from the positive x -axis to the positive y -axis. A triangle is formed by connecting the tip of the line segment to the x -axis perpendicularly (as in the image). What is the maximum area of the triangle?



11. (5 points) Find the largest and smallest values of the function $f(x) = (2x)^{2/3}(-x + 10) + 1$ on the interval $[-4, \frac{1}{2}]$.
12. (5 points) In the figure below, X is moving along the x -axis towards the right at a rate of 2cm/s. Find the rate at which $|PX| + |QX|$, the sum of the distances between X and the points $P(0, 2)$ and $Q(5, 3)$, is changing as X passes the point $(3, 0)$.



13. (3 points) Use the Mean Value Theorem to show that $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ for all $x > 0$. (If it helps, you may use the fact that this is equivalent to proving that $\sqrt[3]{1+x} - (1 + \frac{1}{3}x) < 0$ for all $x > 0$.)
14. (3 points) The position of an object along the x -axis is given by $x(t) = \sin^2(t) - \sin(t)$ for $t \geq 0$.

- (a) Find the velocity function.
(b) Find the distance travelled by the object while $\pi \leq t \leq 2\pi$.

15. (12 points) Evaluate the following integrals.

(a) $\int \left[\frac{2}{3x^2} - \frac{4}{x} + 6^x - \pi^2 \right] dx$

(b) $\int \frac{(2 - \sqrt{x})^2}{2\sqrt{x}} dx$

(c) $\int_{\pi/3}^{\pi/4} \frac{\tan x}{\sec x} dx$

(d) $\int_{-4}^4 (|x| - \sqrt{16 - x^2}) dx$ (Interpret the definite integral as areas.)

16. (3 points) Given $f''(x) = \frac{4}{x^{2/3}} - \frac{3}{x^{1/2}}$, $f'(1) = 12$ and $f(1) = 15$, find $f(x)$.

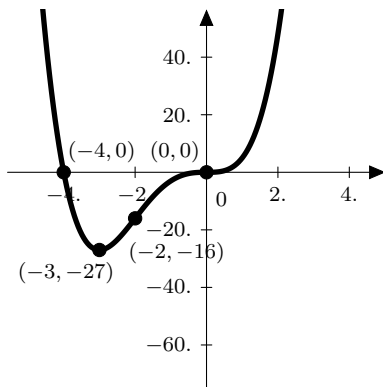
17. (4 points) Compute $\int_0^2 (4x - 3x^2) dx$ as a limit of Riemann Sums. You may find the following identities useful :

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

18. (2 points) Use the Fundamental Theorem of Calculus to find the derivative $f'(x)$ if $f(x) = \int_1^{1/x} \frac{1}{e^{1/t} + 1} dt$. Simplify your answer.

Answers

1. (a) 1 (b) $-\infty$ (c) $-\infty$ (d) -3 (e) 0, 2, 3, 4, 6, 7
2. (a) $\frac{-3}{44}$ (b) -6 (c) π (d) 0 (e) $-\infty$
3. $y = 1$ on the right, and $y = \frac{-1}{5}$ on the left
4. No value of a can make $f(x)$ continuous on \mathbb{R} .
5. (a) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ (b) $\lim_{h \rightarrow 0} \frac{\frac{1}{2(x+h)^2+5} - \frac{1}{2x^2+5}}{h} = \frac{-4x}{(2x^2+5)^2}$
6. (a) $y' = \frac{5\sqrt[3]{x^2}}{3} - \frac{3^x(\ln(3)\ln(5x) - \frac{1}{x})}{[\ln(5x)]^2} + \frac{8\sec(\frac{x}{x})\tan(\frac{x}{x})}{x^2}$ (b) $y' = \frac{ye^x - 2xy\cos(x^2y)}{x^2\cos(x^2y) - e^x}$ (c) $y' = \frac{\tan^2(x)}{(8x^2-7)\sqrt{5x+1}} \left[\frac{2\sec^2(x)}{\tan(x)} - \frac{16x}{8x^2-7} - \frac{5}{2(5x+1)} \right]$ (d) $y' = 9[e^x(x^2-5)+\ln(x)]^8 \left(e^x(x^2+2x-5) + \frac{1}{x} \right)$ (e) $y' = \left[\frac{\csc(x)}{3} \right]^{\log_2(x)} \left(\frac{\ln(\csc(x)) - \ln(3)}{x\ln(2)} - \log_2(x)\cot(x) \right)$
7. $f^{(57)}(x) = xe^x + 57e^x$
8. $x = -1$ (The function is undefined at $x = 3$.)
9. (a) $f'(x) = 4x^2(x+3)$, $f''(x) = 12x(x+2)$ (b) x -intercepts: $x = -4, 0$, y -intercept: $y = 0$ (c) $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$ (d) increasing on $[-3, \infty)$, decreasing on $(-\infty, -3]$ (e) local min: $(-3, -27)$, no local max (f) concave up on $(-\infty, -2]$ and $[0, \infty)$, concave down on $[-2, 0]$ (g) $(-2, -16)$ and $(0, 0)$ (h) See image.



10. 4 units²
11. absolute max: 57, absolute min: 1
12. $\frac{2\sqrt{13}}{13}$ cm/s
13. **OPTION A:** If $f(x) = (1+x)^{1/3}$ then f is differentiable (and thus continuous) on $(-1, \infty)$. For $x > 0$, the mean value theorem applied to f on $[0, x]$ yields a real number x_0 such that $0 < x_0 < x$ and $f(x) = f(0) + f'(x_0)(x-0) = 1 + \frac{1}{3}x(1+x_0)^{-2/3} < 1 + \frac{1}{3}x$, since $(1+x_0)^{-2/3} < 1$.
OR OPTION B: Let $f(x) = \sqrt[3]{1+x} - (1 + \frac{1}{3}x)$, so $f'(x) = \frac{1}{3\sqrt[3]{(1+x)^2}} - \frac{1}{3}$. Note that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. The MVT states that there must exist a point $x_0 \in (0, a)$ such that $f'(x_0) = \frac{f(a)-f(0)}{a-0} = \frac{f(a)}{a}$ for any interval $(0, a)$, which implies that $a \cdot f'(x_0) = f(a)$ must hold for any $a > 0$. However, $f'(x)$ is always negative for $x > 0$, so $a \cdot f'(x_0)$ (and therefore $f(a)$) must always be less than zero when $a > 0$, i.e. $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ for all $x > 0$.
14. (a) $v(t) = 2\sin(t)\cos(t) - \cos(t)$ (b) 4 units (c) $\frac{1-\sqrt{2}}{2}$ (d) $16 - 8\pi$

15. (a) $\frac{-2}{3x} - 4\ln|x| + \frac{6^x}{\ln(6)} - \pi^2x + C$ (b) $4\sqrt{x} - 2x + \frac{\sqrt{x^3}}{3} + C$ (c) $\frac{1-\sqrt{2}}{2}$ (d) $16 - 8\pi$
16. $f(x) = 9\sqrt[3]{x^4} - 4\sqrt{x^3} + 6x + 4$
17. $\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{2}{n} \left[4 \left(\frac{2}{n}i \right) - 3 \left(\frac{2}{n}i \right)^2 \right] \right) = 0$
18. $f'(x) = \frac{-1}{x^2(e^x+1)}$