DDC Final Exam
May 12th, 2017

1. [**2**] Write $z = -2\sqrt{3} + 2i$ in polar form.
2. [**2**] Find the real and imaginary parts of $\frac{1-2i}{3+4i}$.
3. [**2**] Find all complex solutions to $z^3 = -8$.
4. [**6**] In $\mathbb{P}_2$:
   (a) Find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1+t^2, 1-t, -t+t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$.
   (b) Given that $[p]_{\mathcal{B}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, find $[p]_{\mathcal{C}}$ and p.
   (c) Given $q = 1+2t+3t^2$, how would you use matrix multiplication to find $[q]_{\mathcal{B}}$? Just briefly explain what you would do without actually doing it.
5. [**5**] Find the eigenvalues of $A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix}$.
6. [**5**] Let $A = \begin{bmatrix}3 & -5\\1 & -1\end{bmatrix}$. Construct the general solution of $\mathbf{x}' = A\mathbf{x}$ involving complex eigenfunctions.
7. [**5**] Find the least-squares line $y = ax + b$ for the data points $(0,1), (1,3), (2,2), (3,3)$.
8. [**6**] Let $A = \begin{bmatrix}2 & -2 & 18\\2 & 1 & 0\\1 & 2 & 0\end{bmatrix}$.
   (a) Find an orthonormal basis for $ColA$.
   (b) Find a QR factorization of A.
9. [**7**] Consider $\mathbb{P}_2$ together with the inner product $\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$.
   (a) Compute $\|3 - 2t\|$.
   (b) Find the **orthogonal** projection of t^2 onto the subspace spanned by 1 and t.
10. [**6**] Find the second order Fourier approximation to the function

$$f(x) = \begin{cases}0 & 0 \le x < \pi\\ 1 & \pi \le x \le 2\pi\end{cases}$$

    on the interval $[0, 2\pi]$.
11. [**7**] Consider the quadratic form on $\mathbb{R}^2$:

$$Q(\mathbf{x}) = 16x_1^2 - 8x_1x_2 + x_2^2$$

    (a) Find a symmetric matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.
    (b) Make a change of variable $\mathbf{x} = P\mathbf{y}$, P orthogonal, that transforms Q into a quadratic form with no cross-product term. Clearly state P and the new quadratic form.
    (c) Classify Q.
12. [**5**] Find the singular values of $A = \begin{bmatrix}1 & 1\\0 & 1\\1 & 0\end{bmatrix}$.
13. [**5**] A singular value decomposition of A is $A = U\Sigma V^T$ where

$$U = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

$$\Sigma = \begin{bmatrix}5 & 0 & 0\\0 & 3 & 0\end{bmatrix}$$

$$V^T = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{18} & -2/3\\1/\sqrt{2} & 1/\sqrt{18} & 2/3\\0 & -4/\sqrt{18} & 1/3\end{bmatrix}^T$$

    (a) Find a **reduced** singular value decomposition of A.
    (b) Find a **reduced** singular value **expansion** of A.
14. [**5**] Find the minimum polynomial of the given matrix.
    (a) $A = \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$
    (b) $A = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$
    (c) $A = \begin{bmatrix}3&0&0&0&0&0&0\\0&3&0&0&0&0&0\\0&0&3&0&0&0&0\\0&0&0&3&0&0&0\\0&0&0&0&3&0&0\\0&0&0&0&0&3&0\\0&0&0&0&0&0&3\end{bmatrix}$
15. [**6**] In this question H_1, H_2, and H_3 are subspaces of $\mathbb{R}^3$. Show that the sum is direct, or explain why it is not.
    (a) $H_1 + H_2 + H_3$ where $H_1 = span\{\begin{bmatrix}1\\0\\1\end{bmatrix}\}$, $H_2 = span\{\begin{bmatrix}0\\1\\1\end{bmatrix}\}$, $H_3 = span\{\begin{bmatrix}1\\1\\1\end{bmatrix}\}$
    (b) $H_1 + H_2 + H_3$ where
    $H_1 = \{\begin{bmatrix}x\\y\\z\end{bmatrix} : x+y+z = 0\}$,
    $H_2 = \{\begin{bmatrix}x\\y\\z\end{bmatrix} : x-y-z = 0\}$, $H_3 = \{\begin{bmatrix}0\\0\\0\end{bmatrix}\}$
16. [**6**] Find a primary decomposition of $A = \begin{bmatrix}1 & -1 & 1\\-1 & 1 & 1\\0 & 0 & 0\end{bmatrix}$. Hint: $m_A(\lambda) = \lambda^2(\lambda - 2)$.
17. [**2**] Let $z \in \mathbb{C}$. If $|z| = Re(z)$ prove that
    (a) $z \in \mathbb{R}$.
    (b) $z \ge 0$.

18. **[3]** The matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is the change of basis matrix from what basis $\mathcal{B}$ to the basis $\mathcal{C} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$ for $\mathbb{R}^3$?

19. **[4]** Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$. Use diagonalization to find a formula for A^k where k is a positive integer.

20. **[2]** Let A be $n \times n$ and let λ_1 and λ_2 be distinct eigenvalues of A. Show that the intersection of the corresponding eigenspaces is $\{\mathbf{0}\}$.

21. **[3]** For which value(s) of $a \in \mathbb{R}$ is the matrix $A = \begin{bmatrix} 1 & 3 & a \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable?

22. **[3]** Let u and v be vectors in an inner product space, and let $H = span\{u, v\}$. Show that $w \in H^{\perp}$ **if and only if** $\langle w, u \rangle = \langle w, v \rangle = 0$.

23. **[3]** Assume that A is a square matrix with minimum polynomial $m_A(\lambda) = \lambda(\lambda - 2)(\lambda + 3)$.
    (a) Explain why $A^2 + A - 6I \neq 0$.
    (b) Express A^3 as a linear combination of A^2 and A.

ANSWERS

1. $z = 4e^{5\pi i/6}$
2. $-\frac{1}{5}, -\frac{2}{5}$
3. $z = 2e^{i\phi}$, $\phi = \pi/3, \pi, 5\pi/3$
4. (a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$
   (b) $[p]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$, $p = 2 - 2t + 2t^2$
   (c) Calculate $P^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ where P is the matrix found in (a).
5. $\lambda = 0, 1$
6. $\mathbf{x} = c_1 e^{(1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$
7. $y = \frac{1}{2}x + \frac{3}{2}$
8. (a) $\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \}$
   (b) $\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}$
9. (a) $\sqrt{11}$
   (b) $\frac{5}{3} + 2(t-1)$
10. $\frac{1}{2} - \frac{2}{\pi}\sin x$
11. (a) $\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$
    (b) $P = \frac{1}{\sqrt{17}} \begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix}$, $Q' = 17y_1^2$
    (c) Positive semi-definite
12. $\sigma = \sqrt{3}, 1$
13. (a) $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} \\ 1/\sqrt{2} & 1/\sqrt{18} \\ 0 & -4/\sqrt{18} \end{bmatrix}^T$
    (b) $5 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$
    $+ 3 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \end{bmatrix}$
14. (a) $(\lambda - 2)^2$
    (b) λ
    (c) $\lambda - 3$
15. (a) Sum is direct. One should check that all the appropriate intersections are trivial.
    (b) Not direct: $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in H_1 \cap H_2$
16. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}$
17. (a) $|z| = Re(z) \Rightarrow \sqrt{a^2 + b^2} = a \Rightarrow b = 0 \Rightarrow z = a \in \mathbb{R}$
    (b) $z = a = \sqrt{a^2 + b^2} \geq 0$
18. $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \}$
19. $\begin{bmatrix} 2^k & 0 \\ 3^k - 2^k & 3^k \end{bmatrix}$
20. $Ax = \lambda_1 x$ and $Ax = \lambda_2 x \Rightarrow \lambda_1 x = \lambda_2 x \Rightarrow (\lambda_1 - \lambda_2)x = \mathbf{0} \Rightarrow x = \mathbf{0}$ (since $\lambda_1 \neq \lambda_2$)
21. $a = -6$
22. ($\Rightarrow$): $w \in H^{\perp} \Rightarrow \langle w, x \rangle = 0 \ \forall x \in H \Rightarrow \langle w, u \rangle = \langle w, v \rangle = 0$
    ($\Leftarrow$): $\langle w, u \rangle = \langle w, v \rangle = 0 \Rightarrow \forall a, b \in \mathbb{R}$ we have $\langle w, au + bv \rangle = a\langle w, u \rangle + b\langle w, v \rangle = 0 + 0 = 0 \Rightarrow w \in H^{\perp}$
23. (a) The minimum polynomial of A is of degree 3, so no polynomial p of degree 2 can satisfy $p(A) = 0$.
    (b) $m_A(A) = 0 \Rightarrow A(A - 2I)(A + 3I) = 0 \Rightarrow A^3 + A^2 - 6A = 0 \Rightarrow A^3 = -A^2 + 6A$