1. Evaluate the following:

(a) $\displaystyle\int_0^2 \int_{x^2}^4 \sqrt{y}\cos(y^2)\, dy\, dx$

(b) $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z\, dz\, dx\, dy$

2. Evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} \int_0^{3\sec(\theta)} r^3 \sin^2(\theta)\, dr\, d\theta$, by changing to Cartesian coordinates.

3. Combine the sum $\displaystyle\int_0^2 \int_0^x \sqrt{x^2+y^2}\, dy\, dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ into one double integral in polar coordinates. (Do not evaluate the integral).

4. Given $\displaystyle\int_0^{\sqrt{5}} \int_0^{\sqrt{5-x^2}} \int_{x^2+y^2}^5 \sqrt{x^2+y^2}\, dz\, dy\, dx$:

(a) Sketch the solid region over which we are integrating.

(b) Express (do not evaluate) the above integral as:

(i) a triple integral in *cylindrical coordinates*.

(ii) a triple integral in *spherical coordinates*.

5. Let $f(x,y,z) = x - y^3 - 2z^2$. Given a point $P(-4,-2,1)$ on the level surface S defined by $f(x,y,z) = 2$. Find:

(a) an equation of the tangent plane to S at the point P.

(b) the directional derivative of f at P in the direction of $\vec{v} = \langle 3, 6, -2 \rangle$.

(c) the maximum rate of change in f at P.

(d) the tangent line to C at the point P, where C is the curve intersecting S and the plane $2x - 3y - z = -3$.

6. Find and classify the critical points of $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$.

7. Use the method of Lagrange multipliers to find the maximum value of the function $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

8. Find $\dfrac{\partial z}{\partial x}$ for each of the following:

(a) $z = x^4 \cos(x^2 y^3)$,

(b) given $z = \dfrac{\ln(v)}{u^3}$, where $u = x^3 \sin(y)$ and $v = y^2 \cos(x)$.

9. If $z = xy + f(x^2 + y^2)$, show that $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2$.

10. Sketch and name the following:

(a) the polar curve $r = 1 - 2\cos(\theta)$,

(b) the surface $\rho^2 = 1 + 3\rho^2 \cos^2(\phi)$,

(c) the level curve of $f(x,y) = \dfrac{y}{x^2+y^2}$ corresponding to $c = -1$,

(d) the space curve $\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), e^t \rangle$.

11. Let C be the curve with parametric equations: $x = t^3 - 3t$, $y = \dfrac{4}{1+t^2}$.

(a) Find:

(i) the x and y intercepts. (ii) $\dfrac{dy}{dx}$. (iii) all points of horizontal or vertical tangency.

(b) Sketch the graph of C and give its orientation.

(c) Set up (but *do not evaluate*) an integral expression for the area bounded by the loop.

12. Suppose that a curve C given by parametric equations in t passes through the origin when $t = 0$, and satisfies $\dfrac{dx}{dt} = -3\sin(t)\cos^2(t)$, $\dfrac{dy}{dt} = 3\sin^2(t)\cos(t)$.

 Question 12 continues on the next page.

(a) Find the parametric equations for the curve.

(b) Find the length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.

13. Prove that if a particle's speed is constant then its acceleration is directed towards the unit normal vector $\vec{N}$.

14. Compute the curvature of the circular helix $\vec{r}(t) = \langle a\cos(\omega t), a\sin(\omega t), bt\rangle$, with $a > 0$.

15. Given $f(x, y) = \begin{cases} \dfrac{(x+y)^2}{x^2+5y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$, determine the set of all points at which f is continuous.

16. Let $f(x) = \displaystyle\int_0^x \frac{\arctan(t^2)}{t}\, dt$. Find:

(a) the Maclaurin series for $f(x)$.

(b) the radius of convergence of $f(x)$.

(c) an estimate for $f(0.5)$ correctly to within 3 decimal places (justify your calculations).

17. (a) Find the third degree Taylor polynomial $T_3(x)$ for $f(x) = x\ln(x)$ centered at $c = 1$.

(b) Estimate the error in using $T_3(x)$ to approximate $f(x)$ on the closed interval $[0.5, 1.5]$.

18. Find the Taylor series expansion for $f(x) = \dfrac{3}{x^2 - x - 2}$ about $x = 1$. (Hint: partial fractions)

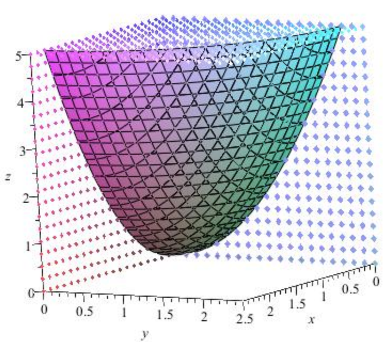
Answers

1. (a) $\frac{1}{2}\sin(16)$ (b) 2π

2. $2\displaystyle\int_0^3 \int_0^x y^2\, dy\, dx = \frac{27}{2}$

3. $\displaystyle\int_0^{\sqrt{8}} \int_0^{\pi/4} r^2\, d\theta\, dr$

4. a) Lower boundary is one quarter of an upright paraboloid, cut off by the xz and yz planes (first octant), upper boundary is the plane $z = 5$.



b) i) $\displaystyle\int_0^{\pi/2} \int_0^{\sqrt{5}} \int_{r^2}^{5} r^2\, dz\, dr\, d\theta$ (order of integration of θ and r can be interchanged)

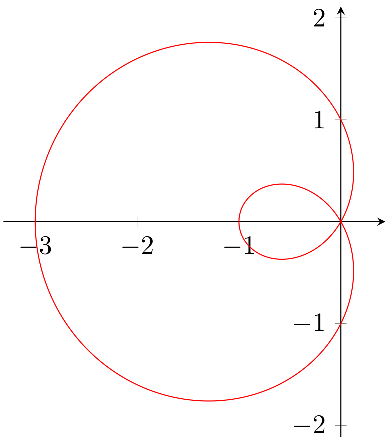
ii) $\displaystyle\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{5\sec\phi} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta + \int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\cot\phi\csc\phi} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta$

5. a) $(x+4) - 12(y+2) - 4(z-1) = 0$ b) $-\frac{61}{7}$ c) $\sqrt{161}$ d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} t + \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

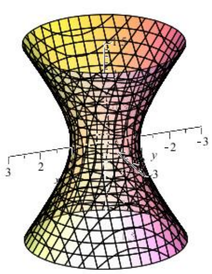
Question 18 continues on the next page.

6. Local minimum at $(0,0)$, local maximum at $(-\frac{5}{3},0)$, saddle points at $(-1,2)$ and $(-1,-2)$.
7. $3+\sqrt{29}$
8. a) $4x^3\cos(x^2y^3) - 2x^5y^3\sin(x^2y^3)$

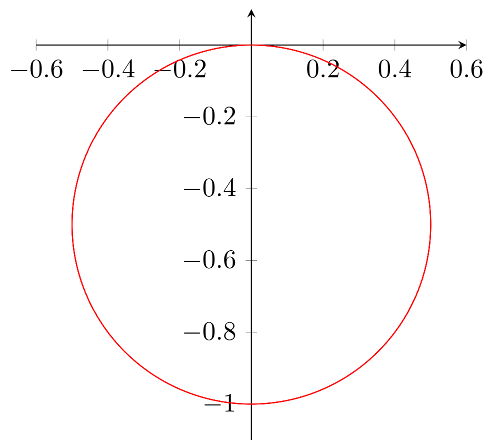
   b) $-\dfrac{3\ln(v)}{u^4}(3x^2\sin(y)) - \dfrac{y^2\sin(x)}{u^3 v}$
9. Let $u = x^2+y^2$, then $\dfrac{\partial z}{\partial x} = y + \dfrac{\partial f}{\partial u}2x$ and $\dfrac{\partial z}{\partial y} = x + \dfrac{\partial f}{\partial u}2y$.
10. a) Limaçon



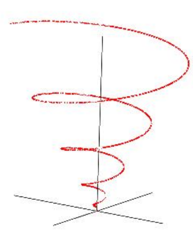
b) Hyperboloid in one sheet



c) Circle of radius $\frac{1}{2}$ centered at $(0,-\frac{1}{2})$.



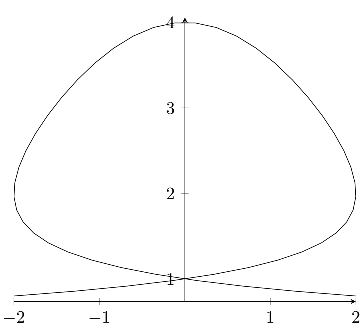
d) A 'helix' of expanding radius, defined only for $z>0$.



11. a) i) $(0,4)$, $(0,1)$, no $x-$intercepts. ii) $-\dfrac{8t}{(1+t^2)^2(3t^2-3)}$

   iii) Horizontal tangent at $(0,4)$, vertical tangent at $(-2,2)$ and $(2,2)$.

   b)The graph is oriented counterclockwise.



   c) $\displaystyle\int_{\sqrt{3}}^{-\sqrt{3}} \frac{12(t^2-1)}{t^2+1}\,dt$

Question 18 continues on the next page.

12. a) $x = \cos^3(t) - 1$, $y = \sin^3(t)$ b) $\frac{3}{2}$ units

13. If v is constant then $\dfrac{dv}{dt} = 0$, so there is no tangential component of acceleration.

14. $\kappa = \dfrac{a\omega^2}{a^2\omega^2 + b^2}$

15. $\mathbb{R}^2 \setminus \{(0,0)\}$

16. a) $\displaystyle\sum_{n=0}^{\infty} 2(-1)^n \frac{x^{4n+2}}{(4n+2)^2}$ b) 1 c) 0.124

17. a) $(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$ b) $\frac{1}{24}$

18. $\displaystyle\sum_{n=0}^{\infty} -(x-1)^n \left(1 + \frac{(-1)^n}{2^{n+1}}\right)$