1. (8 points) Solve each of the following systems, or show that it is inconsistent, as appropriate.

(a) $\begin{cases} 3x + 2y = 3 \\ -9x - 4y + 2z = 0 \\ 6x + 2y - z = 1 \end{cases}$

(b) $\begin{cases} 2x_1 - 6x_2 + 4x_3 + 2x_4 + 4x_5 = 4 \\ -3x_1 + 9x_2 - 6x_3 - 6x_4 = -27 \end{cases}$

2. (6 points) For the system $\begin{cases} x + 7y + kz = 1 \\ 2x + 15y + z = 3 \\ x + 7y + k^2 z = k \end{cases}$ Find the value(s) of k, if any, for which the system has

(a) a unique solution

(b) infinitely many solutions

(c) no solution

3. (5 points) Given $A = \begin{bmatrix} 4 & 0 \\ -5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -6 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 5 & b^2 - 9 & 1 \\ -5 & a & 6 \\ 1 & 6 & 3 - a \end{bmatrix}$.

Find the following, or indicate that it is not possible, as appropriate:

(a) $B^T A^T$

(b) $(AB)^T$

(c) $2A^2 + BC$

(d) The value(s) of a and b, if any, so that D is symmetric.

4. (4 points) A neighbour wants to start a company that creates shoes using only recycled materials (tires, plastic bottles and rope fiber). She designed three types of shoes that require different amounts of material: a single running shoe requires 80 grams of tire, 100 grams of plastic bottle and 70 grams of rope fiber. A single dress shoe requires 100 grams of tire, 70 grams of plastic bottle and 60 grams of rope fiber. A single sandal requires 120 grams of tire, 40 grams of plastic bottle and 50 grams of rope fiber. She has on hand 960 grams of tire, 760 grams of plastic bottles and 620 grams of rope fiber. She has asked you to help her determine how many prototypes of each kind of shoe can she make provided that she **only makes pairs of shoes**.

(a) Define your variables and set up the system needed to determine the solution.
Do not solve the system.

(b) Given that the general solution to the system is $\{2+t, 8-2t, t\}$, find all realistic solutions to this problem.

5. (3 points) Let A and B be $n \times n$ matrices.
Are the following statements True or False? Briefly justify your answer.

(a) If $\det(A) = 0$, then the system $AX = B$ has no solution.

(b) If both $\det(A)$ and $\det(B)$ are non-zero, AB is an invertible matrix.

(c) If $A + B$ is an invertible matrix, then $\det(A) \neq 0$.

6. (6 points) Let A, B, and C be 3×3 matrices. Assume A is not invertible, $\det(B) = \frac{1}{2}$, and $\det(C) = -3$. Find the following, or state that there is not enough information.

(a) $\det(2B^{-1}C^2)$

(b) $\det(A^TB + A^TC)$

(c) $\det(B + C)$

(d) $\text{rank}(B)$

7. (2 points) Find $\det(A) = \begin{vmatrix} 1 & 2 & 6 & 8 & 4 \\ 1 & -3 & 13 & -5 & 1 \\ 0 & 0 & -2 & 3 & 5 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

8. (4 points) Given $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & k \\ 1 & -1 & -5 \end{bmatrix}$, find all the value(s) of k, if any, so that:

(a) rank of A equals 2

(b) nullity of A equals 0

9. (3 points) Consider the linear system $\begin{cases} x_1 + 4x_2 - 3x_3 + 2x_4 = 6 \\ x_2 - x_3 = 9 \\ x_1 + 4x_2 + 2x_4 = 3 \\ 3x_1 + 10x_2 + x_3 + 2x_4 = 1 \end{cases}$

Given that $det(A) = \begin{vmatrix} 1 & 4 & -3 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 4 & 0 & 2 \\ 3 & 10 & 1 & 2 \end{vmatrix} = -12$, use Cramer's Rule to solve for x_2 **only**.

10. (2 points) Consider the planes $\mathcal{P}_1$: $4x + 2y + z = 13$ and $\mathcal{P}_2$: $3x - 7y + 2z = 15$.

(a) Find the normal vectors for both planes.

(b) Determine if the planes are parallel, perpendicular, or neither.

11. (6 points) Given the points $P = (3, 1, 2), Q = (-1, 0, 4)$, and $R = (7, -2, 5)$.

(a) Find $\|\overrightarrow{PQ}\|$

(b) Find a vector equation for the line parallel to $\overrightarrow{QR}$ that is passing through P.

(c) Is the point $D = (11, -1, 3)$ on the line?

(d) Find a general equation (in form $ax + by + cz = d$) for the plane going through the points P, Q, and R.

 Question 11 continues on the next page.

(e) Is that plane a subspace of $\mathbb{R}^3$? Justify your answer.

12. (4 points) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y \leq 0 \right\}$

(a) Is $\mathbf{0}$ in S? Justify.

(b) Give two nonzero vectors in S.

(c) Is S a subspace of $\mathbb{R}^3$? Justify.

13. (7 points) Given the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, and $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Find Span$\{\mathbf{u}, \mathbf{v}\}$. If a line, express in vector form. If a plane, express in general form.

(b) Find Span$\{\mathbf{u}, \mathbf{w}\}$. If a line, express in vector form. If a plane, express in general form.

(c) Find Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. If a line, express in vector form. If a plane, express in general form.

(d) Is $\mathbf{0}$ in Span$\{\mathbf{u}\}$? Justify.

(e) Selecting only from these four vectors, list all possible bases for $\mathbb{R}^3$ (if any).

14. (9 points) Suppose a matrix $U = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} & \mathbf{u_4} & \mathbf{u_5} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 5 & 1 & a & 0 & 5 \\ 4 & 1 & 9 & b & 5 \end{bmatrix}$

reduces to $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) For each set, determine whether it is linearly independent or dependent. Justify.
(i) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$
(ii) $\{\mathbf{u_1}, \mathbf{u_3}, \mathbf{u_4}\}$
(iii) $\{\mathbf{u_2}, \mathbf{u_3}, \mathbf{u_5}\}$
(iv) $\{\mathbf{u_4}, \mathbf{u_5}\}$
(v) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$

(b) List the sets in (a) that could serve as a basis for Col(U).

(c) What is Col(U)? Circle the best description below:
A point   A line   A plane $\mathbb{R}^3$

(d) Find the values of the constants a and b located in the matrix U.

(e) Find a basis for Nul(U) and state its dimension.

(f) For what value of m is Nul(U) a subspace of $\mathbb{R}^m$?

15. (3 points) Given that $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = 4x \right\}$ is a subspace of $\mathbb{R}^3$.,

 Question 15 continues on the next page.

(a) Find a basis for S.

(b) What is the dimension of S?

16. (3 points) For parts (a) and (b), suppose that A is a $4 \times n$ matrix for which the nullity of A^T is 1.

(a) What is the rank of A?

(b) If the nullity of A is 3, what is the value of n?

17. (5 points) An economy consists of two industries: Aerospace and Nuclear Power. Let production of $1 in Aerospace require 30 cents in Aerospace and 20 cents in Nuclear Power. Let production of $1 in Nuclear Power require 70 cents in Aerospace and 40 cents in Nuclear Power. There is an external demand of $42 million in Aerospace and $28 million in Nuclear Power.

(a) Which, if any, of the industries are profitable? Justify your answer.

(b) How much of each should be produced in order to meet the external demand?

(c) What is the internal consumption?

18. (6 points) Using the graphical method, find the values for x_1 and x_2 which maximize the objective function $z = x_1 + 1.5x_2$ subject to the following constraints:

$$\begin{cases} x_1 + x_2 & \leq & 8 \\ x_1 + 2x_2 & \leq & 12 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{cases}$$

19. (6 points) In order to increase wellness and productivity at work, the company Acti-link has recently invested in desks with adjustable height so that employees can either sit or stand at their workstation, and easily switch between the two positions. The company observed that if an employee is in the standing position at a given time, there is a 80% probability that the employee will be sitting one hour later. If the employee is sitting at a given time, there is a 30% chance that the employee will change position one hour later.

(a) Find a transition matrix P associated with this situation.

(b) Paul likes to start his working day standing at his desk. What is the probability that he will be sitting two hours after the beginning of his shift?

(c) Find a steady-state vector associated with the matrix P found in part (a). Your answer should be given using fractions.

20. (8 points) A thief enters a shop and threatens the clerk, forcing him to open the safe. The clerk says, ‘The code for the safe is changed every day, and if you hurt me you’ll never get the code”. The code was encrypted using the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$ in a Hill 2-cipher, and the result is: **DGRUCPTF**

(a) Find the decryption matrix A^{-1}.
For easy reference, here are the reciprocals modulo 26:

 Question 20 continues on the next page.

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

(b) Use matrix multiplication to verify your answer to part (a).

(c) Decode the message the thief used to open the safe (which he claims he got from the clerk). For easy reference, here are the numbers corresponding to each letter of the alphabet:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

ANSWERS

1. (a) $\{x = 2/3, y = 1/2, z = 4\}$ (b) $\{x_1 = -5 + 3r - 2s - 4t, x_2 = r, x_3 = s, x_4 = 7 + 2t, x_5 = t\}$

2. (a) $k \neq 0, 1$ (b) $k = 1$ (c) $k = 0$

3. (a) $\begin{bmatrix} 8 & -10 \\ 0 & 15 \\ 16 & -29 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & -10 \\ 0 & 15 \\ 16 & -29 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 2 \\ -37 & 28 \end{bmatrix}$ (d) $b = \pm 2$, $a \in \mathbb{R}$

4. (a) $\left\{ \begin{array}{rcrcrcr} 80x & + & 100y & + & 120z & = & 960 \\ 100x & + & 70y & + & 40z & = & 760 \\ 70x & + & 60y & + & 50z & = & 620 \end{array} \right\}$

(x : number of running shoes, y : number of dress shoes, z : number of sandals)

(b) $\{x = 2, y = 8, z = 0\}$ or $\{x = 4, y = 4, z = 2\}$ or $\{x = 6, y = 0, z = 4\}$

5. (a) False, the system may also have infinitely many solutions depending on B.

(b) True, $det(AB) = det(A)det(B) \neq 0$

(c) False, if B is invertible and $A = O$ then $A + B$ is invertible but $det(A) = 0$

6. (a) 144 (b) 0 (c) not enough info (d) 3

7.$|A| = -80$

8. (a) $k = -1$ (b) $k \neq -1$

9. $x_2 = \frac{-96}{-12} = 8$

 Question 20 continues on the next page.

10. (a) $\mathbf{n_1} = (4, 2, 1)$, $\mathbf{n_2} = (3, -7, 2)$ (b) the planes are perpendicular since $\mathbf{n_1} \cdot \mathbf{n_2} = 0$

11. (a) $\sqrt{21}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix}$ (c) yes ($t = 1$) (d) $3x + 20y + 16z = 61$
(e) no (not through $\mathbf{0}$)

12. (a) yes (b) answers vary (c) No. Not closed under Scalar Multiplication.

13. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$ (b) $x + 2y - z = 0$ (c) $x + 2y - z = 0$ (d) yes $\mathbf{0} = 0\mathbf{u}$
(e) No basis for $\mathbb{R}^3$ can be made from among these four.

14. (a) LD, LI, LI, LI, LD (b) (ii) and (iii) (c) $\mathbb{R}^3$ (d) $a = 11$, $b = 1$ (e) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$,
dimension $= 2$ (f) $m = 5$

15. One possible basis is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, dimension $= 2$.

16. (a) 3 (b) $n = 6$

17. (a) Aerospace is profitable $(0.3 + 0.2 < 1)$ but Nuclear Power is not $(0.7 + 0.4 \geq 1)$

(b) \$160 million in Aerospace and \$100 million in Nuclear Power

(c) \$118 million in Aerospace and \$72 million in Nuclear Power

18. (4,4)

19. (a) $P = \begin{bmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{bmatrix}$ (b) 72% (c) $\begin{bmatrix} 3/11 \\ 8/11 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 21 \\ 7 & 12 \end{bmatrix}$ (b) $AA^{-1} = I$ (c) CHANGED