1. You are given matrix $A = \begin{bmatrix} 0 & 0 & 2 & 4 & 0 \\ 2 & 2 & 0 & 6 & 4 \\ 1 & 1 & 2 & 7 & 2 \end{bmatrix}$ and the reduced row echelon form of A is $\begin{bmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) (3 points) Give the general solution for $A\mathbf{x} = \mathbf{0}$

(b) (1 point) Find the dimension of Nul A.

(c) (1 point) Find a basis for Col A.

(d) (2 points) For what value of a is $\begin{bmatrix} 6 \\ a \\ 0 \end{bmatrix}$ in Col A?

(e) (3 points) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ be the columns of A.
For each of the following sets state whether the set **is** or **is not** a basis for Col A? Give a brief justification for your answers.

i. $\{\mathbf{a}_2, \mathbf{a}_4\}$

ii. $\{\mathbf{a}_2, \mathbf{a}_5\}$

iii. $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

2. Given the following system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ x_1 + x_2 + ax_3 &= 6 \\ x_1 + 4x_2 - x_3 &= b \end{aligned}$$

(a) (1 point) For what values of a and b is this system inconsistent?

(b) (1 point) For what values of a and b does this system have a unique solution?

(c) (1 point) For what values of a and b does this system have infinitely many solutions?

(d) (2 points) For what values of a and b is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ a solution of this system?

3. (4 points) Use row reduction to find a function of the form $y = a_0 + a_1x + a_2x^2$ that passes through the points

$$(-1, 1), (1, 5), (2, 1)$$

4.

(a) (1 point) In $\mathbb{R}^2$ find a vector, $\mathbf{u}$, parallel to the line $y = -2x$.

(b) (1 point) In $\mathbb{R}^2$ find a vector, $\mathbf{v}$, perpendicular to the line $y = -2x$.

(c) (2 points) Write the vector $\begin{bmatrix} 5 \\ 15 \end{bmatrix}$ as a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$ that you found above.

5. Let $A = \begin{bmatrix} 3 & a \\ 1 & b \end{bmatrix}$.

(a) (2 points) For what values of a and b (if any) is A symmetric?

(b) (2 points) For what values of a and b (if any) is $A^2 = A$?

(c) (2 points) For what values of a and b (if any) is $A = A^{-1}$?

(d) (2 points) Find a condition on a and b such that A invertible?

6. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(a) (2 points) Compute A^2.

(b) (2 points) Based on the answer in part (a) what is A^{-1}?

7. Let $S(\mathbf{x}) = A\mathbf{x}$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects vectors through the y-axis, and let $R(\mathbf{x}) = B\mathbf{x}$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects vectors through the line $y = x$.

(a) (3 points) Find the standard matrix of the transformation $T = R \circ S$.

(b) (1 point) Find the angle of rotation corresponding to transformation T.

8. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear transformation such that

$$\begin{aligned} T(\mathbf{a} + \mathbf{b}) &= \mathbf{c} \\ T(\mathbf{a} + \mathbf{c}) &= \mathbf{b} \\ T(\mathbf{b} + \mathbf{c}) &= \mathbf{a} \end{aligned}$$

(a) (2 points) What is $T^{-1}(\mathbf{a})$?

(b) (2 points) What is $T(\mathbf{a} + \mathbf{b} + \mathbf{c})$ in simplest form?
Hint: start by adding the three given equations together.

(c) (2 points) What is $T(\mathbf{a})$?

9. (4 points) Write $\begin{bmatrix} 1 & 1 \\ 1 & 100 \end{bmatrix}$ as a product of elementary matrices.

10. (4 points) Suppose A and B are invertible $n \times n$ matrices. Find the inverse of $\begin{bmatrix} A & I \\ B^{-1}A & 0 \end{bmatrix}$

11. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 7 \end{bmatrix}$.

(a) (3 points) Evaluate the determinant of A.

(b) (2 points) Use Cramer's Rule to solve for x_4 **only** in the system $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$.

12. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Let $V = \{\mathbf{x} \in \mathbb{R}^2 \ : \ \mathbf{x} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{v}\}$

(a) (3 points) Find a basis for V, given that V is a subspace of $\mathbb{R}^2$.

(b) (1 point) Draw the subspace V.

13. Suppose A and B are $n \times n$ matrices and let $V = \{X \in M_{n\times n} \ : \ AXB = BXA\}$

(a) (4 points) Show that V is a subspace of $M_{n\times n}$.

(b) (2 points) If A is invertible and V is defined as above show that A^{-1} will be in V.

14. Let $S = \{1 + 2x - x^2, 1 + 3x^2, 4x + ax^2\}$

(a) (3 points) For what value(s) of a is S linearly dependent?

(b) (1 point) When $a = 0$ what is the dimension of the span of S?

15. Let $\mathcal{L}_1$ be the line $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and let $\mathcal{L}_2$ be the line $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(a) (3 points) Find the distance between the parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) (3 points) Find an equation for the line through the origin that intersects $\mathcal{L}_1$ at a right angle.

16. Given $A(1, 1, 1)$, $B(2, 2, 2)$, and $C(3, -1, 2)$

(a) (3 points) Find an equation of the form $ax + by + cz = d$ for the plane containing A, B, and C.

(b) (2 points) Find the area of triangle ABC.

(c) (2 points) Find the cosine of the angle θ at vertex A of triangle ABC.

(d) (2 points) Find the point on side AC that is 2 units from A.

17. Let $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

(a) (2 points) Compute $\mathbf{e}_1 \times \mathbf{w}$, $\mathbf{e}_2 \times \mathbf{w}$, and $\mathbf{e}_3 \times \mathbf{w}$.

(b) (3 points) Now use the above to evaluate and simplify

$$\|\mathbf{e}_1 \times \mathbf{w}\|^2 + \|\mathbf{e}_2 \times \mathbf{w}\|^2 + \|\mathbf{e}_3 \times \mathbf{w}\|^2$$

Your final answer should be expressed in terms of $\|\mathbf{w}\|$.

18. (8 points) Fill in the correct numerical value for each of of the following statements.

(a) Suppose that A is a 7×4 matrix of rank 3 and that $A\mathbf{x} = \mathbf{b}$ is consistent then the rank of the 7×5 augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is _____.

(b) Suppose that $\left\{ \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ a \\ b \end{bmatrix} \right\}$ is linearly dependent then $a =$ _____ and $b =$ _____.

(c) Suppose $\det(A) = 5$ and $\det(2A) = 40$ then $\det(3A) =$ _____.

(d) Suppose A is a 6×6 matrix and that Row A, Col A, and Nul A all have the same dimension then the rank of A is _____.

ANSWERS

1. (a) $\mathbf{x} = r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (b) 3 (c) $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$ (d) $a = -12$

2. (a) $b \neq -3$ (b) $b = -3, a \neq 5$ (c) $b = -3, a = 5$ (d) $b = -3, a = 5$

3. $y = 5 - 2x + 2x^2$

4. (a) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 5 \\ 15 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

5. (a) $a = 1$, $b =$ any value (b) $a = -6, b = -2$ (c) $a = -8, b = -3$ (d) $a \neq -3b$

6. (a) $4I$ (b) $\frac{1}{4}A$

7. (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) 90°

8. (a) $\mathbf{b} + \mathbf{c}$ (b) $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (c) $\frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

9. For example, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 99 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 0 & A^{-1}B \\ I & -B \end{bmatrix}$

11. (a) 8 (b) $x_4 = -1$

12. (a) $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$

13. (b) Substituting $X = A^{-1}$ gives $AXB = AA^{-1}B = B$ and $BXA = BA^{-1}A = B$.

14. (a) $a = -8$ (b) 3

15. (a) $\sqrt{11}$ (b) $\mathbf{x} = t \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$

16. (a) $3x_y - 4z = 0$ (b) $\frac{\sqrt{26}}{2}$ (c) $\frac{1}{3\sqrt{3}}$ (d) $\begin{bmatrix} 7/3 \\ -1/3 \\ 5/3 \end{bmatrix}$

17. (b) $2\|\mathbf{w}\|^2$

18. (a) 3 (b) $a = 4/3, b = -14/3$ (c) 135 (d) 3