

1. (12 points) Evaluate the following limits.

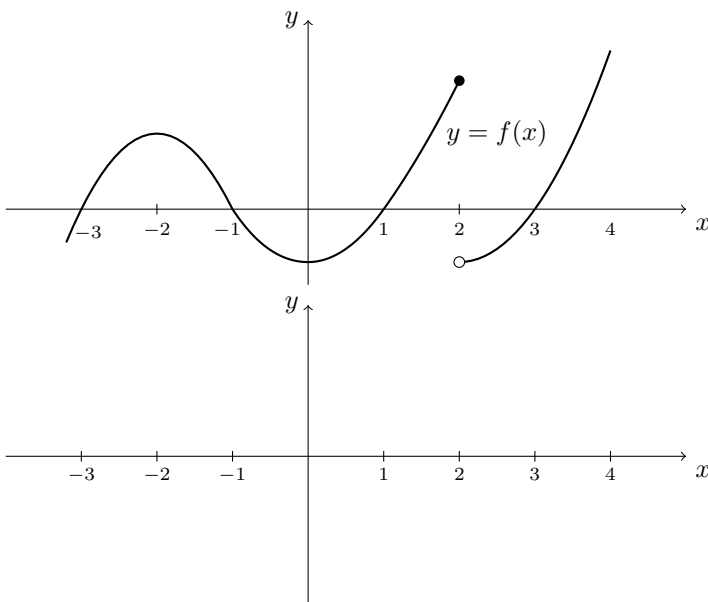
Use $-\infty$, ∞ or “does not exist”, wherever appropriate.

- (a) $\lim_{x \rightarrow 3} \frac{\frac{3}{7} - \frac{x}{3x-2}}{x-3}$
- (b) $\lim_{x \rightarrow -3^+} \frac{x^2 - 9}{x^2 + 6x + 9}$
- (c) $\lim_{x \rightarrow 0} \frac{x + \sin x}{\tan x}$
- (d) $\lim_{x \rightarrow 0} \frac{2 \cos^2 x - 7 \cos x + 5}{\cos x - 1}$
- (e) $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1} - (7 - 2x)}{x - 2}$
- (f) $\lim_{x \rightarrow \infty} \frac{2x + \sin x}{3x}$

2. Consider the function

$$f(x) = \begin{cases} \frac{x+3}{x^2+7x+12} & \text{if } x < -3, \\ x^2+3 & \text{if } -3 \leq x < -2 \text{ and} \\ \frac{x+9}{x+3} & \text{if } x > -2. \end{cases}$$

- (a) (4 points) Identify all points of discontinuity, and state whether the discontinuity is removable, jump or infinite.
- (b) (2 points) Find all horizontal asymptotes of f .
3. Consider the function $f(x) = \sqrt{x^2 - 5}$.
- (a) (3 points) Find $f'(x)$ by using the limit definition of the derivative.
- (b) (2 points) Find an equation of the tangent line to the curve $y = f(x)$ at $x = 3$.
4. (2 points) Given the following graph of f , draw a rough sketch of a graph of its derivative f' on the given axes.



5. (15 points) Find the derivative $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

- (a) $y = 1 + 2x^3 + \frac{4}{\sqrt[5]{x}} + 6^x + \tan\left(\frac{7}{8}\pi\right) + 9 \log_{10}(x)$
- (b) $y = \sec(3x^2 + 2) \cos(8xe^x)$
- (c) $y = \ln\left(\sqrt[5]{\frac{(2x^3 + 1) \sin x}{(4x - 1)^6}}\right)$
- (d) $\sqrt{4x^2 + 3y^3} = x + y$
- (e) $y = \frac{5x}{1 + x^{\sin x}}$ (Hint: what is $\frac{d}{dx}(x^{\sin x})$?)

6. (2 points) Show that if f , g and h are differentiable, and h is not zero, then $\left(\frac{fg}{h}\right)' = \frac{f'gh + fg'h - fgh'}{h^2}$.

7. (2 points) Find the 81st derivative of $f(x) = \cos(10x)$.

8. (3 points) For what value(s) of x in the interval $[0, 2\pi]$ does the curve $y = e^x \cos x$ have a horizontal tangent?

9. (5 points) Barbara is flying a kite in a large field. The kite is 50 feet above the ground and moves horizontally away from Barbara at a speed of 4 ft/s. At what rate is the angle between the string and the horizontal changing when 100 feet of string have been let out?

10. (4 points) Show that the equation $\cos(3x) + 2016x = 0$ has exactly one solution.

11. (4 points) Find the absolute maximum and minimum values of $f(x) = (x^2 + 2x)^{2/3}$ on the interval $[-3, 2]$.

12. Given $f(x) = \frac{(x-1)^2}{(x+1)^2}$, $f'(x) = \frac{4(x-1)}{(x+1)^3}$

and $f''(x) = \frac{8(2-x)}{(x+1)^4}$;

- (a) (1 point) state the domain of f ;
- (b) (2 points) find the intervals on which f is increasing and decreasing;
- (c) (1 point) find all local maxima and minima of f ;
- (d) (2 points) find the intervals on which f is concave up and concave down;
- (e) (1 point) find any points of inflection of f .

13. (5 points) You are given the following information about a function f :

- f is continuous everywhere except at -1 .
- x -intercepts: $-2, 1, 3$
- y -intercept: -1
- $f(2) = 2, f(4) = 2$
- f has a vertical asymptote at $x = -1$
- $\lim_{x \rightarrow -\infty} f(x) = -3, \lim_{x \rightarrow \infty} f(x) = 4$
- $f'(x) > 0$ on $(-\infty, -1) \cup (-1, 2) \cup (3, \infty)$
- $f'(x) < 0$ on $(2, 3)$
- $f''(x) > 0$ on $(-\infty, -1) \cup (1, 2) \cup (2, 4)$
- $f''(x) < 0$ on $(-1, 1) \cup 4, \infty)$

Use all of the above information to sketch a graph of f . Clearly label all asymptotes, local extrema and points of inflection.

14. (5 points) Find the point(s) on the curve $y = x^2$ that are closest to the point $(0, 3)$. Justify your answer.

15. (3 points) Find the function f satisfying

$$f''(x) = x^2 + \sin x - 2e^x + 3, f'(0) = 1 \text{ and } f(0) = 4.$$

16. (12 points) Evaluate the following integrals.

(a) $\int \left(\frac{\sqrt{x^5}}{x^3} - e^x - \cos 3 \right) dx$

(b) $\int_1^e \frac{x^2 + 2x + 1}{x^3 + x^2} dx$

(c) $\int (\sec^2 x)(1 + \sin x) dx$

(d) $\int_0^4 |2x - 3| dx$

17. (2 points) Evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1^5 + 2^5 + 3^5 + \dots + n^5}{n^5} \right)$ by expressing it as a definite integral.

18. Given

$$g(x) = \int_2^{\sqrt{x}} \frac{t}{\ln(1+t)} dt,$$

find:

- (a) (1 point) $g(4)$
- (b) (2 points) $g'(x)$

19. (2 points) Given that

$$\int_1^3 f(x) dx = 8, \quad \int_2^5 f(x) dx = -3,$$

$$\text{and } \int_1^2 f(x) dx = \int_2^3 f(x) dx,$$

find:

- (a) $\int_2^3 f(x) dx;$
- (b) $\int_1^5 f(x) dx.$

20. (1 point) If $\lim_{x \rightarrow 4} f(x) = 0$ and $\lim_{x \rightarrow 4} \frac{f(x)}{g(x) - \pi} = 10$, then what is $\lim_{x \rightarrow 4} g(x)$?

Answers

1. (a) $\frac{2}{49}$

(b) $-\infty$

(c) 2

(d) -3

(e) 1

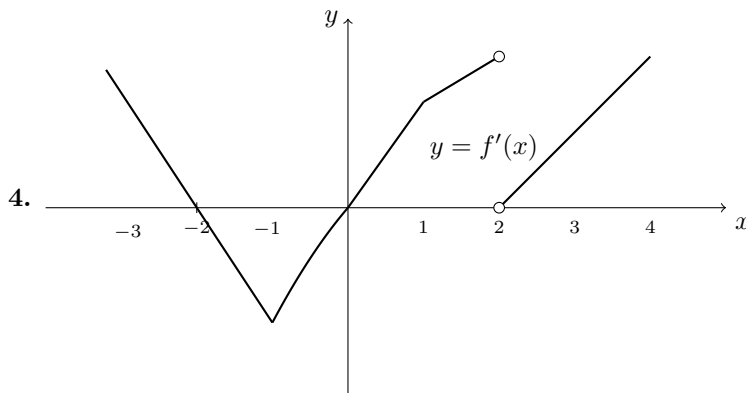
(f) $\frac{2}{3}$

2. (a) $x = -4$ (infinite), $x = -3$ (jump),
 $x = -2$ (removable)

(b) $y = 0$ (at $-\infty$) and $y = 1$ (at $+\infty$)

3. (a) $f'(x) = \frac{x}{\sqrt{x^2 - 5}}$

(b) $y - 2 = \frac{3}{2}(x - 3)$



5. (a) $y' = 6x^2 - \frac{4}{5}x^{-6/5} + 6^x \ln 6 + \frac{9}{x \ln 10}$
- (b) $y' = 6x \sec(3x^2 + 1) \tan(3x^2 + 1) \cos(8xe^x) - \frac{8e^x(x+1)\sin(8xe^x)}{\cos(3x^2 + 2)}$
- (c) $y' = \frac{1}{5} \left(\frac{6x^2}{2x^3 + 1} + \cot x - \frac{24}{4x - 1} \right)$
- (d) $y' = -\frac{8x - 2(x+y)}{9y^2 - 2(x+y)}$ (Simplified)
- (e) $y' = \frac{5(1 + x^{\sin x}) - 5x \cdot x^{\sin x} \left(\ln x \cos x + \frac{\sin x}{x} \right)}{(1 + x^{\sin x})^2}$

6.

$$\left(\frac{fg}{h} \right)' = \frac{(fg)'h - fgh'}{h^2} = \frac{(f'g + fg')h - fgh'}{h^2} = \frac{f'gh + fg'h - fgh'}{h^2}$$

7. As $f^{(4)}(x) = 10^4 \cos 10x$, we have that $f^{(80)}(x) = 10^{80} \cos 10x$ and so $f^{(81)}(x) = -10^{81} \sin 10x$

8. $f'(x) = e^x(\cos x - \sin x)$, so $f'(x) = 0$ if $\sin x = \cos x$, which happens at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ in the interval $[0, 2\pi]$

9. If Barbara is x feet from the point on the ground directly beneath the kite and θ is the angle of elevation of the string, then $x = 50 \cot \theta$, so $\frac{dx}{dt} = -50 \csc^2 \theta \frac{d\theta}{dt}$, or equivalently, $\frac{d\theta}{dt} = -\frac{1}{50} \sin^2 \theta \frac{dx}{dt}$

Now $\frac{dx}{dt} = 4$, and when 100 feet of string have been released, $\sin \theta = \frac{50}{100} = \frac{1}{2}$, so at this instant the angle of elevation of the string is decreasing at a rate of $\frac{1}{50} \cdot \left(\frac{1}{2}\right)^2 \cdot 4 = \frac{1}{50}$ radians per second.

10. Let $f(x) = \cos(3x) + 2016x$. This function is continuous and differentiable everywhere.

In particular, the intermediate value theorem will apply to any closed interval. Letting $x = 0$, $f(0) = 1$, and letting $x = -1$, $f(-1) = \cos(-3) - 2016$. Clearly $-2017 \leq f(-1) \leq -2015$, so $f(-1)$ is negative. Applying the IVT on the interval $[-1, 0]$, we find that f must have a root in this interval, that is, there must be a number c , with $-1 < c < 0$, such that $f(c) = 0$, since $f(-1) < 0 < f(0)$.

Suppose now that there are *two* roots c and d , and suppose (without loss of generality) that $c < d$. Then as $f(c) = f(d) = 0$, and f is continuous and differentiable everywhere, we can apply Rolle's theorem on the interval $[c, d]$. We find that there must be a number r in the interval (c, d) such that $f'(r) = 0$.

However, computing f' , we find that $f'(x) = -3 \sin(3x) + 2016 > 0$ for any x . Therefore, by contradiction, there cannot be two roots f .

This shows that c is the only root of f .

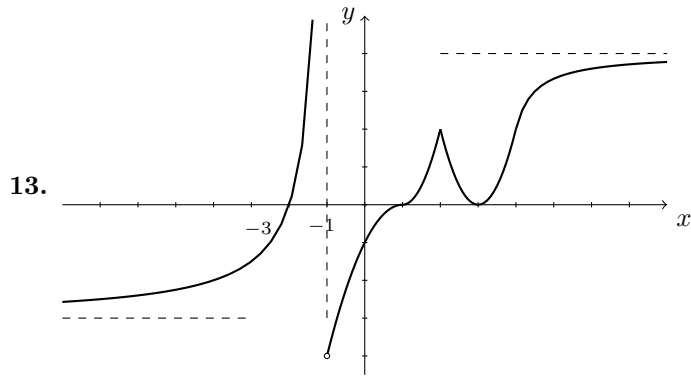
11. $f'(x) = \frac{4(x+1)}{3(x^2+2x)^{1/3}}$. The critical numbers of f are $x = -1$, $x = 0$ and $x = -2$

Calculating the value of f and the critical numbers and the endpoints, we find:

$$f(-3) = \sqrt[3]{9}, f(-2) = 0, f(-1) = 1, f(0) = 0 \text{ and } f(2) = 4$$

Thus the absolute maximum value is 4, attained at $x = 2$, and the absolute minimum value is 0, attained at $x = -2$ and $x = 0$.

12. (a) $\mathbb{R} \setminus \{-1\}$, or $(-\infty, -1) \cup (-1, \infty)$
- (b) f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$
- (c) Local minimum at $x = 1$, at the point $(1, 0)$
- (d) f is CU on $(-\infty, -1) \cup (-1, 2)$ and CD on $(2, \infty)$
- (e) There is one inflection point at $(2, \frac{1}{9})$



14. The points are $(\pm\sqrt{\frac{5}{2}}, \frac{5}{2})$ with minimum distance $\frac{\sqrt{11}}{2}$

15. $f(x) = \frac{1}{12}x^4 - \sin x - 2e^x + \frac{3}{2}x^2 + 4x + 6$

16. (a) $2\sqrt{x} - e^x - (\cos 3)x + C$

(b) $2 - \frac{1}{e}$

(c) $\tan x + \sec x + C$

(d) $\frac{17}{2}$

17. This limit is $\int_0^1 x^5 dx = \frac{1}{6}$

18. (a) $g(4) = 0$

(b) $g'(x) = \frac{1}{2 \ln(1 + \sqrt{x})}$

19. (a) $\int_2^3 f(x) dx = 4$

(b) $\int_1^5 f(x) dx = 1$

20. If $\lim_{x \rightarrow 4} g(x) \neq \pi$, then $\lim_{x \rightarrow 4} \frac{f(x)}{g(x) - \pi} = 0$. Since this limit is given to be 10, it must be that $\lim_{x \rightarrow 4} g(x) = \pi$