1. (8 points) Solve each of the following systems, or show that it is inconsistent, as appropriate.

(a) $$\begin{cases} 3x + y + 2z = 11 \\ y - z = -1 \\ -2x + 3y - z = 1 \\ 5x + y + z = 10 \end{cases}$$

(b) $$\begin{cases} 3x_1 - 3x_2 + 2x_3 - 4x_4 = -4 \\ 2x_1 - 2x_2 + x_3 - 2x_4 = -3 \\ -3x_1 + 3x_2 - x_3 + 2x_4 = 5 \end{cases}$$

2. (6 points) For the system $\begin{cases} x + 3y - 2z = 0 \\ x + (k+2)y - z = -2 \\ 2x + 6y + kz = 2 \end{cases}$ Find the value(s) of k, if any, for which the system has

(a) infinitely many solutions

(b) no solution

(c) a unique solution

3. (5 points) Given $A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 4 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} s-6 & t^2 \\ 3t+4 & s \end{bmatrix}$, and $D = \begin{bmatrix} 1 & -2 \\ 3 & -9 \end{bmatrix}$.

Find (if possible)

(a) $(AB)^T$

(b) The value(s) of s and t, if any, so that C is symmetric.

(c) A 2×2 matrix E such that $DE = I$.

(d) A vector $\mathbf{x}$ such that $D^{-1}\mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

4. (1 point) TRUE or FALSE? For all matrices A and B for which the product AB is defined, $(AB)^T = A^T B^T$. Justify your answer.

5. (3 points) The Acme Novelty company manufactures dodads, gizmos, widgets, and whatnots at their factory. Each of these products uses a number of snaps, screws, and switches, as shown in the table below:

	Snaps	Screws	Switches
Dodads	1	2	4
Gizmos	2	5	7
Widgets	1	2	3
Whatnots	1	3	3

They have on hand 9 snaps, 21 screws, and 29 switches, and they want to know the number of each product that should be produced in order to use up all of these materials.

(a) Define your variables and set up the system needed to determine the solution. Do not solve the system.

 Question 5 continues on the next page.

(b) Given that the general solution to the system is $\{t-1, 3-t, 4, t\}$, find all realistic solutions to this problem.

6. (4 points) Let A and B be 4×4 matrices with $\det(A) = -2$ and $\det(B) = 3$.
Find each determinant or state that there is not enough information.

(a) $\det\left((A^T B)^{-1}\right)$

(b) $\det\left(2A^2(B)^{-1}\right)$

7. (1 point) True or False? If A and B are $n \times n$ matrices, then $\det(-A) = -\det(A)$. Briefly justify your answer.

8. (3 points) Find $\det(A) = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 3 & 2 & 1 \\ 0 & 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{vmatrix}$

9. (4 points) Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & -2 & -4 \\ 3 & -1 & -2 \end{bmatrix}$. Find A^{-1} using the adjoint matrix.

10. (5 points) If $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4$, find

(a) $\det(B) = \begin{vmatrix} a & 2d & 3g \\ b & 2e & 3h \\ c & 2f & 3i \end{vmatrix}$

(b) $\det(C) = \begin{vmatrix} -3a & -3b & -3c \\ 2a-d & 2b-e & 2c-f \\ g & h & i \end{vmatrix}$

(c) $\det(A + C)$

11. (4 points) Given the points $A(3, 5, 1)$, $B(5, 4, 4)$ and $C(10, 8, 2)$.

(a) Find $\overrightarrow{AB} \times \overrightarrow{BC}$.

(b) Find a general form equation ($ax + by + cz = d$) for the plane containing the points A, B, and C.

(c) Show that the triangle ABC is a right triangle. (Hint: the right angle is at vertex B).

12. (5 points) Consider the points $A(-3, 4)$ and $B(1, -2)$.

(a) On the grid provided, sketch the vector $\overrightarrow{AB}$ with its initial point located at the origin.

(b) On the same grid, sketch the vector $\frac{1}{2}\overrightarrow{AB}$ with its initial point at A.

(c) Find an equation in vector form for the line which passes through A and is parallel to $\overrightarrow{AB}$.

(d) Find $\|\overrightarrow{AB}\|$.

 Question 12 continues on the next page.

(e) Find a unit vector which is oppositely-directed to $\overrightarrow{AB}$.

13. (1 point) Given the two planes: $\mathcal{P}_1$: $3x - y + 2z = 4$ and $\mathcal{P}_2$: $kx + 2y - 4z = 5$ For what value(s) of k will $\mathcal{P}_1$ and $\mathcal{P}_2$ be parallel?

14. (5 points) For the planes $\left\{\begin{array}{rcrcrcl} 3x & + & 4y & - & z & = & 2 \\ x & + & 2y & + & 5z & = & 1 \\ 7x & + & 10y & + & 3z & = & c \end{array}\right.$

(a) For what value(s) of c do the three planes have a common intersection?

(b) Assuming that the planes do intersect, describe their intersection. Is it a point, a line, or a plane?

15. (3 points)

(a) Suppose $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\} = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Determine whether the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is linearly dependent or linearly independent, or indicate that there is not enough information, as appropriate.

(b) Suppose $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a set of vectors in $\mathbb{R}^2$. Determine whether the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent or linearly independent, or indicate that there is not enough information, as appropriate.

(c) Suppose $\det(A) = 6$ for some 3×3 matrix A. Determine whether the set formed by the columns of A is linearly dependent or linearly independent, or indicate that there is not enough information, as appropriate.

16. (6 points) Let $\mathbf{v_1} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -5 \\ 4 \\ 4 \end{pmatrix}$, $\mathbf{v_3} = \begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix}$, $\mathbf{v_4} = \begin{pmatrix} -7 \\ a \\ 5 \end{pmatrix}$.

(a) Find all value(s) of a for which $\mathbf{v_4}$ is in $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

(b) Find all value(s) of a for which $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\} = \mathbb{R}^3$

(c) Describe $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ geometrically. (Is it a point, a line, a plane, all of $\mathbb{R}^3$?)

(d) What is the dimension of $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$?

17. (3 points) Let A be an 8×5 matrix with nullity 3.

(a) How many solutions does $AX = 0$ have?

(b) Find the nullity of A^T.

18. (8 points) Suppose a matrix $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} & \mathbf{a_5} & \mathbf{a_6} \end{bmatrix} = \begin{bmatrix} 1 & 3 & -3 & -4 & -4 & 0 \\ 2 & 6 & -1 & 12 & 1 & 0 \\ -1 & -3 & 0 & -8 & -2 & 0 \\ 3 & 9 & -4 & 8 & -5 & 0 \end{bmatrix}$

reduces to $R = \begin{bmatrix} 1 & 3 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Give a basis for $\text{Col}(A)$.

 Question 18 continues on the next page.

(b) Give a basis for Nul(A).

(c) Indicate whether each of the following sets of columns from A are linearly independent or not: (i) $\{\mathbf{a_1}, \mathbf{a_4}, \mathbf{a_5}\}$ (ii) $\{\mathbf{a_6}\}$

(d) Write $\mathbf{a_3}$ as a linear combination of the column vectors $\mathbf{a_1}$ and $\mathbf{a_4}$.

19. (4 points) Let $S = \{(x, y, z)\,|3(x-2) + 2(y-1) - 4(z-2) = 0\}$

(a) Is $\mathbf{0} \in S$?

(b) Give two nonzero vectors from S.

(c) Is S a subspace? Justify your answer.

20. (5 points) A simple economy of robot creation has two industries: steel and silicon (for computer chips). In order for robots to manufacture \$1 of steel, it takes 10¢ steel and 20¢ of silicon (to run the equipment). While to manufacture \$1 of silicon it takes 70¢ steel and 40¢ of silicon.

(a) Give the consumption matrix C associated with this economy.

(b) Determine whether or not each of the industries is profitable. Justify your answer.

(c) Determine whether if the economy is productive. Justify your answer.

(d) Determine the production necessary to satisfy an external demand of \$2000 of steel and \$1200 of silicon.

21. (6 points) Using the graphical method find possible values for x and y that minimize the z when $z = -6x + 2y$. Subject to the following constraints.

$$\begin{cases} 9x \ -3y \leq 63 \\ 3x \quad +y \leq 39 \\ 10x +2y \geq 40 \\ x \qquad \geq 0 \\ y \qquad \geq 0 \end{cases}$$

Show your work.

22. (6 points) Generally, if a passenger is in the same airport as their luggage, there is a 95% probability that their bag will be placed on the correct flight and that both the passenger and their luggage will be in the same airport once the plane has landed. The rest of the time, then passenger's bag is placed on the wrong plane, so the passenger and their luggage end up landing in different airports.

However, if a passenger ever ends up in a different airport than their luggage, there is a 2% probability that the mistake will be quickly corrected, and that the passenger's bag will end up in the same airport as them after their next flight.

(a) Find a transition matrix P associated with this situation.

(b) Bernard has just arrived at the Dorval airport with his luggage. He is flying to Atlanta, then getting on a connecting flight to Dallas. What is the probability that his luggage will also be in Dallas when he lands?

 Question 22 continues on the next page.

(c) Find a steady-state vector associated with the matrix P found in part (a). Your answer should be given using fractions.

23. (4 points) A message is encoded using the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$ in a Hill 2-cipher, and the result is as follows:

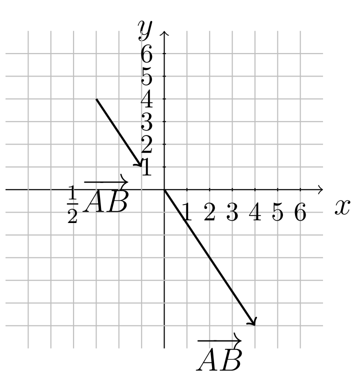
QK UB WT

(a) Use matrix multiplication to show that $A^{-1} = \begin{bmatrix} 15 & 23 \\ 23 & 6 \end{bmatrix}$ is the decryption matrix.

(b) Are you skilled enough to decode the two-word message?

ANSWERS

1. (a) $\{x = 1, y = 2, z = 3\}$ (b) $\{x_1 = -2 + s, x_2 = s, x_3 = 1 + 2t, x_4 = t\}$
2. (a) impossible (b) $k = -4$ or 1 (c) $k \neq 1, -4$
3. (a) $\begin{bmatrix} 5 & -2 \\ -8 & 6 \end{bmatrix}$ (b) $t = -1$ or 4 (s can be anything) (c) $\begin{bmatrix} 3 & -2/3 \\ 1 & -1/3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ -12 \end{bmatrix}$
4. False
5. (a) $\left\{ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 9 \\ 2x_1 & + & 5x_2 & + & 2x_3 & + & 3x_4 & = & 21 \\ 4x_1 & + & 7x_2 & + & 3x_3 & + & 3x_4 & = & 29 \end{array} \right\}$ (x_1 : number of dodads, x_2 : number of gizmos, x_3 : number of widgets, and x_4 : number of whatnots) (b) $\{x_1 = 0, x_2 = 2, x_3 = 4, x_4 = 1\}$ or $\{x_1 = 1, x_2 = 1, x_3 = 4, x_4 = 2\}$ or $\{x_1 = 2, x_2 = 0, x_3 = 4, x_4 = 3\}$
6. (a) $-1/6$ (b) $64/3$ 7. False 8. -720 9. $\frac{-1}{5}\begin{bmatrix} 0 & 1 & -2 \\ -10 & -17 & 19 \\ 5 & 10 & -10 \end{bmatrix}$
10. (a) -24 (b) -12 (c) 0
11. (a) $\begin{bmatrix} -10 \\ 19 \\ 13 \end{bmatrix}$ (b) $-10x + 19y + 13z = 78$ (c) $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$
12. (a) and (b) below (c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 4 \\ -6 \end{bmatrix}$ (d) $2\sqrt{13}$ (e) $\begin{bmatrix} -2\sqrt{13}/13 \\ 3\sqrt{13}/13 \end{bmatrix}$



13. $k = -6$ 14. (a) $c = 5$ (b) a line 15. (a) L.D. (b) L.D. (c) L.I.
16. (a) $a = 17$ (b) $a \neq 17$ (c) a plane (d) 2

 Question 23 continues on the next page.

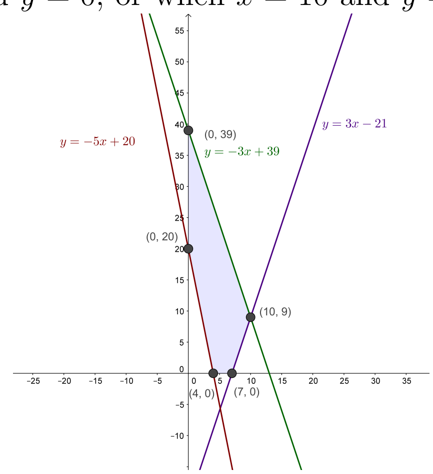
17. (a) infinitely many (b) 6

18. (a) $\{\mathbf{a_1}, \mathbf{a_3}, \mathbf{a_5}\}$ (b) $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (c) (i) L.I. (ii) L.D. (d) $\mathbf{a_3} = -2\mathbf{a_1} + \frac{1}{4}\mathbf{a_4}$

19. (a) Yes (b) $(2, 1, 2)$ and $(4, 2, 4)$ (many answers possible) (c) Yes, S is a 2-dimensional span (provide a possible basis).

20. (a) $C = \begin{bmatrix} 0.1 & 0.7 \\ 0.2 & 0.4 \end{bmatrix}$ (b) The steel industry is profitable, the silicon industry is not. (c) Yes. (various justifications) (d) \$5 100 in steel and \$3 700 in silicon

21. Minimum $z = -42$ occurs when $x = 7$ and $y = 0$, or when $x = 10$ and $y = 9$, or at any of the points



on the line segment connecting $(7, 0)$ to $(10, 9)$

22. (a) $P = \begin{bmatrix} 0.95 & 0.02 \\ 0.05 & 0.98 \end{bmatrix}$ (b) 90.35% (c) $\begin{bmatrix} 2/7 \\ 5/7 \end{bmatrix}$

23. (a) $AA^{-1} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (mod 26) (b) NO WAY