(7) 1. Given below is the the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$.

$$[A|\mathbf{b}] = \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 1 & 1 \\ 2 & 5 & 3 & 0 & 1 \\ -1 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & 2 & 1 \end{array}\right]$$

(a) Determine whether $\mathbf{x} = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ is a solution to the system.

(b) Find the general solution of this system in parametric-vector form.

(c) What is the general solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$?

(d) Write the fourth column of A as a linear combination of the first three columns of A.

(5) 2. Let

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{bmatrix}$$

(a) For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have no solution ?

(b) For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have a unique solution ?

(c) For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions ?

(5) 3. Use matrices to find the quadratic polynomial whose graph goes through the points $(-1, 3)$, $(0, 3)$ and $(1, 7)$.

(3) 4. Find the inverse of $A = \begin{bmatrix} 1 & -1 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

(6) 5. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$.

(a) Evaluate $A^T A$ and find $\left(A^T A\right)^{-1}$.

(b) Evaluate AA^T and show that AA^T is not invertible.

(3) 6. Find an LU-factorization for the matrix $\begin{bmatrix} -2 & -2 \\ -4 & -1 \\ -10 & 2 \end{bmatrix}$.

(5) 7. Let $A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$.

(a) Apply an elementary row operation to A such that the resulting matrix is a **lower triangular** matrix.

(b) Find the elementary matrix that corresponds to the row operation from part (a).

(c) Use the above to find an upper triangular matrix U and a lower triangular matrix L such that $A = UL$. (Note this is not the same as LU-factorization of A.)

(6) 8. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and let $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}$.

(b) Find a basis for the range of T.

(c) Find a basis for the kernel of T.

(d) Is T onto? Is T one-to-one?

(6) 9. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) Find a 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ is a rotation and $T(\mathbf{u})$ is orthogonal to $\mathbf{u}$.

(b) Find a 2×2 matrix B such that $S(\mathbf{x}) = B\mathbf{x}$ is a horizontal shear and $S(\mathbf{u})$ is orthogonal to $\mathbf{u}$.

(c) Draw $\mathbf{u}$ and $T(S(\mathbf{u}))$.

(4) 10. Expand and simplify

$$\begin{bmatrix} -A^{-1} & BA^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & B \\ A & 0 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ B^{-1} & A \end{bmatrix}$$

(5) 11. Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix}$

(a) Find $\det A$.

(b) What is $\det(-2A^{-1}A^TA)$?

(4) 12. Suppose A, B and C are $n \times n$ matrices such that $ABCA = I$.

(a) Use determinants to explain why A, B and C are invertible.

(b) Find C^{-1} in terms of A and B (in simplest form).

(6) 13. Find the **rank** and **nullity** (dimension of null space) of each matrix A described below.

(a) A is a 5×5 elementary matrix.

(b) A is a matrix such that $T(\mathbf{x}) = A\mathbf{x}$ is an onto transformation from $\mathbb{R}^7$ to $\mathbb{R}^5$.

(c) A is a **non-zero** 2×2 matrix such that A^2 is the **zero** matrix.

(5) 14. Consider the set $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : abcd = 0 \right\}$.

(a) Is H closed under scalar multiplication? Justify your answer.

(b) Is H closed under addition? Justify your answer.

(6) 15. Let $V = \{p(x) \in \mathbb{P}_2 : p(0) = -p'(1)\}$.

(a) Find a basis for V.

(b) For what value of k is $p(x) = 6x + k$ in V?

(c) For the polynomial $p(x)$ in part (b), is $p'(x)$ in V? Is $p''(x)$ in V?

(8) 16. Consider the lines $\mathcal{L}_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -4 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathcal{L}_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 9 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

(a) Find the coordinates of the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Let $\mathcal{P}$ be the plane that contains the point $Q(2, 1, 1)$ and is orthogonal to the line $\mathcal{L}_1$. Give the equation (in $ax + by + cz = d$ form) of this plane.

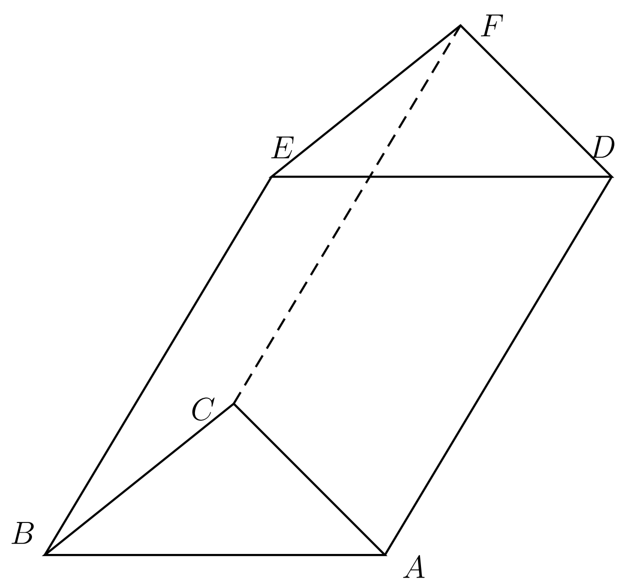
(c) Find the cosine of the angle between $\mathcal{L}_1$ and $\mathcal{L}_2$.

(7) 17. Consider the prism in $\mathbb{R}^3$ (Note that a prism can be seen as half a parallelepiped.) whose triangular base has vertices at the points $A(0, 1, 3)$, $B(2, -1, 3)$, and $C(1, 1, 5)$. Furthermore assume that another vertex of this prism is at $D(4, 7, 10)$. (See the image below).

(a) Find a parametric vector equation for the line through A and B.

(b) Find the area of triangle $\triangle ABC$.

(c) Find the volume of the prism. (Note that $\overrightarrow{AD}$ is not necessarily orthogonal to $\triangle ABC$.)



(3) 18. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be unit vectors in $\mathbb{R}^n$. Furthermore, let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be orthogonal to each other. Simplify the following.

$$\text{Proj}_{\mathbf{u}+\mathbf{w}}(\mathbf{u} - 2\mathbf{v})$$

(2) 19. Suppose A is an $m \times n$ matrix and that there is a matrix C such that $AC = I$. Show that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^m$. What can you conclude about the rank of A?

(4) 20. Fill in the blanks. The missing word is **might**, **must** or **cannot**.

(a) If $A^2 + 3A = 2I$, then A __________ be invertible.

(b) If $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ for three given points A, B, and C in $\mathbb{R}^n$, then Span$\{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}\}$ __________ be three-dimensional.

(c) Two lines in $\mathbb{R}^3$ that are orthogonal to a third line __________ be parallel.

(d) If $\mathbf{a}, \mathbf{2a} + \mathbf{3b}, \mathbf{a} - \mathbf{3c}$ are linearly independent vectors in a vector space V, then $\mathbf{a}, \mathbf{b}, \mathbf{c}$ __________ be linearly independent.