Department of Mathematics
Final Examination

16 May 2016
14h–17h

Mathematical Models
201-225-AB

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Student name: ____________________

Student number: ____________________

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Instructions

1. Do not open this booklet before the examination begins.
2. Check that this booklet contains 6 pages, excluding this cover page and the formula sheet.
3. Write all of your solutions in this booklet and show all supporting work.
4. If the space provided is not sufficient, continue the solution on the opposite page.

(4) **1.** Use Newton's method to find the root of $f(x) = x^3 - 3x^2 + 3$ that is between 2 and 3. Give your answer accurate to 3 decimals.

2. Find y'. **Do not simplify your answer.**

(3) (a) $y = \arcsin(\sqrt{x})$

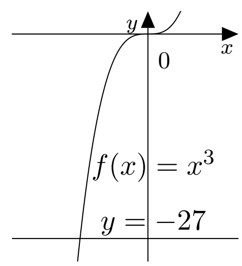
(3) (b) $y = (\arctan(e^x))^3$

(3) (c) $y = \sec^{-1}(3x^2 + 1)$

(3) (d) $\tan(x^{-1}y) = x^2 e^y$ Hint: Solve for y'.

(4) **3.** Use Trapezoidal Rule to approximate $\int_3^6 \sin\big(\cos(x)\big)\, dx$, using $n = 6$ (give your answer to 3 decimals)

(3) **4.** Find the area of the region enclosed by $y = x^3$, the y-axis and $y = -27$. See figure at right.



(4) **5.** The volume and radius of a cylinder are increasing at a rate of 50π cm^3/s and 2 cm/s respectively. At what rate is the height of the cylinder changing dh/dt, when the volume is 36π cm^3 and the radius is 3 cm? (Recall: $V = \pi r^2 h$) Hint: use implicit differentiation.

6. Given $f(x) = \dfrac{(x-2)(2x-1)}{(x+1)^2}$ $f'(x) = \dfrac{9(x-1)}{(x+1)^3}$ and $f''(x) = \dfrac{18(2-x)}{(x+1)^4}$

Find (if any):

(1) (a) The x and y intercept(s).

(1) (b) The vertical and horizontal asymptotes.

(1) (c) The intervals on which f is increasing or decreasing

(1) (d) The local (relative) maxima and minima

(1) (e) The inflection points.

(1) (f) Intervals of upward or downward concavity.

(4) (g) Sketch the graph of f.

7. Evaluate the following limits.

(3) (a) $\displaystyle\lim_{x \to 2} \frac{\cos(x-2) + 2x - 5}{x - 4 + 2e^{x-2}}$

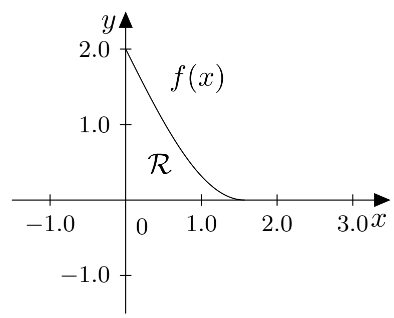
(3) (b) $\lim_{x\to 1} \left(x\right)^{\frac{1}{x-1}}$

(3) (c) $\lim_{x\to 0} x\csc(x)$

(3) **8.** If a resistor of R omhs is connected across a battery of E volts with internal resistor r omhs, then the power (in watts) in the external resistor is given by $P = \dfrac{E^2R}{(R+r)^2}$. If E and r are constants such that $E = 16$V and $r = 8\Omega$, but R varies, what is the maximum possible value of P? Hint: Find dP/dR.

(3) **9.** Determine if $y = e^x\sin(2x) + 4$ is a solution to the differential equation $y'' - 2y' + 5y = 20$.

10. Let $\mathcal{R}$ be the region bounded by the functions $f(x) = 2 - 2\sin x$, $y = 0$ and $0 \le x \le \dfrac{\pi}{2}$. Set up (**but do not evaluate**) the integrals to find the volume of the solid of revolution obtained by revolving $\mathcal{R}$ about;



(2) (a) the $y-$axis

(2) (b) the $x-$axis

(2) (c) the line $y = -1$

(4) **11.** Solve the following differential equation for y. $(\ln y)\frac{dy}{dx} - xy = 0$; with initial condition $y(\sqrt{2}) = e$

12. Integrate the following integrals.

(4) (a) $\displaystyle\int e^x\cos x\,dx$

(4) (b) $\displaystyle\int \cos^2 x\,dx$

(4) (c) $\displaystyle\int \frac{x^3}{x^2+1}\,dx$

(4) (d) $\displaystyle\int \sqrt{x^2-4x+7}\,dx$

(4) (e) $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$

(4) (f) $\displaystyle\int \sin^5 x\cos^3 x\,dx$

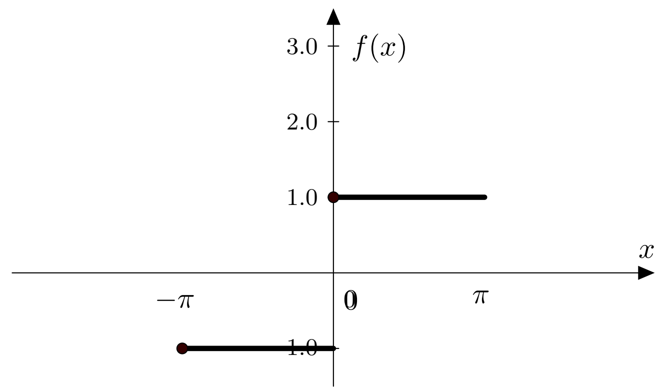
(4) (g) $\displaystyle\int \frac{x+5}{x^3+2x^2-3x}\,dx$

(4) **13.** Solve the following first order linear differential equation for y.

$$\frac{dy}{dx} + (\sec x)y = \cos x \qquad \text{with initial condition } x = 0 \text{ when } y = 5/2$$

14. Given the function and its graph.

$$f(x) = \begin{cases} -1, & \text{if } -\pi \le x < 0 \\ 1, & \text{if } 0 \le x < \pi \end{cases}$$



(2) (a) Determine if the function is even or odd. Show your work or Explain to obtain full marks

(4) (b) Find the first three non-zero terms of the Fourier series for the function above and write the function expansion.

$y - y_0 \approx f'(x_0)(x - x_0)$; $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$; $\displaystyle\int_a^b f(x)\ dx = F(b) - F(a)$; $\displaystyle\int u dv = uv - \int v du$

$y_{rms} = \sqrt{\dfrac{1}{T}\displaystyle\int_0^T y^2\ dx}$; $V_C = \dfrac{1}{C}\displaystyle\int i\ dt$; $s = \displaystyle\int v\ dt$; $v = \displaystyle\int a\ dt$; $q = \displaystyle\int i\ dt$

$$\int_a^b f(x)\ dx \approx \left(\frac{b-a}{2n}\right)\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\int_a^b f(x)\ dx \approx \left(\frac{b-a}{3n}\right)\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

$V = \displaystyle\int_a^b \pi\left([\text{outer radius}]^2 - [\text{inner radius}]^2\right) dx$; $V = \displaystyle\int_a^b 2\pi\ [\text{radius}] \times [\text{height}]\ dx$

$\csc(x) = \dfrac{1}{\sin(x)}$; $\sec(x) = \dfrac{1}{\cos(x)}$; $\cot(x) = \dfrac{1}{\tan(x)}$; $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$; $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

$\sin^2(x) + \cos^2(x) = 1$; $1 + \tan^2(x) = \sec^2(x)$; $1 + \cot^2(x) = \csc^2(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$; $\sin(2x) = 2\sin(x)\cos(x)$; $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$; $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$

$\displaystyle\int \tan(x)\ dx = \ln\left|\sec(x)\right| + C$; $\displaystyle\int \sec(x)\ dx = \ln\left|\sec(x) + \tan(x)\right| + C$

$\displaystyle\int \cot(x)\ dx = \ln\left|\sin(x)\right| + C$; $\displaystyle\int \csc(x)\ dx = \ln\left|\csc(x) - \cot(x)\right| + C$

$\displaystyle\int \sec^n(x)\ dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\ dx$ (for $n > 2$)

$\sqrt{a^2 - x^2} \rightarrow$ sub $x = a\sin(\theta)$; $\sqrt{a^2 + x^2} \rightarrow$ sub $x = a\tan(\theta)$; $\sqrt{x^2 - a^2} \rightarrow$ sub $x = a\sec(\theta)$

$\dfrac{dy}{dx} + P(x)y = Q(x) \quad\longrightarrow\quad y = e^{-\int P(x)dx}\displaystyle\int Q(x)\ e^{\int P(x)dx}\ dx$

$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots + b_1\sin(x) + b_2\sin(2x) + \cdots + b_n\sin(nx) + \cdots$

$a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\ dx$; $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx$; $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx$

Answers

1. 2.532

2. (a) $y' = \dfrac{1}{2\sqrt{x(1-x)}}$

(b) $y' = \dfrac{3e^x \arctan^2(e^x)}{1+e^{2x}}$

(c) $y' = \dfrac{6x}{(3x^2+1)\sqrt{(3x^2+1)^2-1}}$

(d) $y' = \dfrac{2xe^y + x^{-2}y\sec^2(x^{-1}y)}{\sec^2(x^{-1}y)x^{-1} - x^2e^y}$

3. -0.350

4. 60.75

5. 2/9

6. (a) x-int $= 2$, x -int $= 1/2$ and y-int $= 2$

(b) V.A. $x = -1$; H.A. $y = 2$

(c) Increasing: $-\infty < x < -1$ and $1 < x < \infty$
Decreasing: $-1 < x < 1$

(d) Local min: $x = 1$

(e) I.P. $x = 2$

(f) C.U. $]-\infty, -1[\cup]-1, 2[$
C.D. $]2, \infty[$

7. (a) 2/3

(b) e

(c) 1

8. 8 watts

9. yes

10. (a) Shell: $V = 2\pi \displaystyle\int_0^{\pi/2} x(2 - 2\sin x)\, dx$

(b) Disk: $V = \pi \displaystyle\int_0^{\pi/2} (2 - 2\sin x)^2\, dx$

(c) Washer: $V = \pi \displaystyle\int_0^{\pi/2} [(3 - 2\sin x)^2 - 1]\, dx$

11. $y = e^{\sqrt{x^2-1}}$

12. (a) $\dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + C$

(b) $\dfrac{1}{2}e^x(\cos x + \sin x) + C$

(c) $\dfrac{1}{2}(x^2 - \ln(x^2+1)) + C$

(d) $\frac{1}{2}(x-2)\sqrt{x^2-4x+7}+\frac{3}{2}\ln\left(\frac{\sqrt{x^2-4x+7}+x-2}{\sqrt{3}}\right)+C$

(e) $\ln x(\ln(\ln x)-1)+C$

(f) $\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+C$

(g) $-\frac{5}{3}\ln x+\frac{1}{6}\ln(x+3)+\frac{3}{2}\ln(x-1)+C$

13. $\frac{x-\cos x+7/2}{\sec x+\tan x}$

14. (a) $f(x)$ is odd

(b) coefficients: $b_1 = 4/\pi; b_3 = 4/\ 3\pi; b_5 = 4/5\pi$

function: $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(nx)$