1. (3 points) Solve the following system or show that it is inconsistent. $\begin{cases} 3x - 2y + z = 8 \\ -3x + 3y - 2z = -13 \\ -6x + 7y + 4z = 14 \end{cases}$

2. (4 points) Given $\begin{cases} x + hy = k \\ 2x + 4y = 7 \end{cases}$ For what value(s) of h and k does the system have

(a) one solution?

(b) no solutions?

(c) infinitely many solutions?

3. (5 points) David makes plush animals from rags. A bat requires 20 grams of stuffing and 2 rags. A giant snake requires 240 grams of stuffing and 26 rags. A kangaroo requires 80 grams of stuffing and 9 rags. David has 1560 grams of stuffing and 174 rags, and he would like to use them all up. How many bats, snakes and kangaroos should he make? List all realistic possibilities.

4. (4 points) Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ -5 & 1 \end{bmatrix}$. Find, or identify as undefined:

(a) A^{-1}

(b) $(CA)^T$

(c) $A^2 + 3BB^{-1}$

5. (3 points) Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & 5 \\ 7 & k \end{bmatrix}$. For what value(s) of k is the product AB symmetric?

6. (3 points) Let A, B, C and X be $n \times n$ matrices. Below, someone was trying to solve for X in the matrix equation $((A+X)B)^{-1} = C$ but made one or more mistakes! Identify the mistake(s) and then solve for X correctly. (You may assume that any necessary matrices are invertible.)

Equation to solve for X: $((A+X)B)^{-1} = C$

Incorrect solution:

$$\begin{aligned} ((A+X)B)^{-1} &= C \\ B^{-1}(A+X)^{-1} &= C \\ (A+X)^{-1} &= BC \\ A^{-1} + X^{-1} &= BC \\ X^{-1} &= BC - A^{-1} \\ X &= (BC - A^{-1})^{-1} \end{aligned}$$

Corrected solution:

7. (3 points) $A = \begin{bmatrix} 0 & 4 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 3 & 0 \end{bmatrix}$. Find A^{-1} or show that it does not exist.

8. (3 points) Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, evaluate the following determinants:

(a) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

(b) $\begin{vmatrix} a & b & c \\ 2d+3a & 2e+3b & 2f+3c \\ -g & -h & -i \end{vmatrix}$

9. (3 points) If A is a 5×5 matrix with $\det(A) = -2$,

(a) Find the determinant of $10A^T$.

(b) Find the determinant of $\operatorname{adj}(A)$.

10. Given: $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 4 \\ -9 \end{bmatrix}$,

(a) (3 points) Find $\operatorname{adj}(A)$

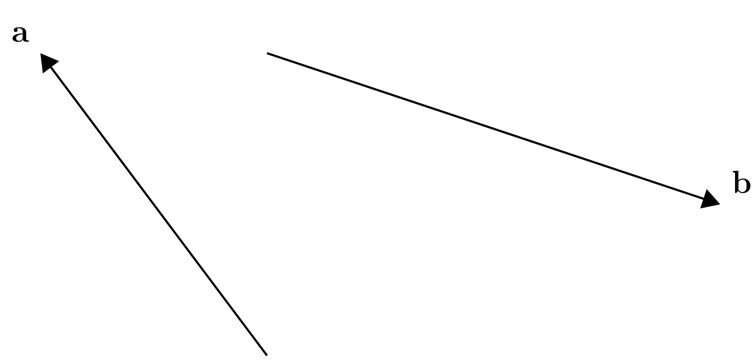
(b) (2 points) Compute $\det(A)$

(c) (1 point) Find A^{-1}

(d) (2 points) Use A^{-1} to solve the system $AX = B$

11. (4 points) Use Cramer's Rule to solve for z in the system: $\begin{cases} x + 5y + 5z = 0 \\ 4x + 7y + 6z = 2 \\ 2x + 4y + 9z = 0 \end{cases}$

12. (1 point) Given **a** and **b** below, draw (and clearly label) the vector $\frac{1}{2}\mathbf{a} + \mathbf{b}$.



13. (3 points) Find equations for the plane $2x + 3y - z = 5$ in parametric form.

14. (5 points) Let $A = (3, -2, 5)$ and $B = (-1, 4, 1)$

 Question 14 continues on the next page.

(a) Find $\overrightarrow{AB}$.

(b) Find a parametric vector equation for the line parallel to $\overrightarrow{AB}$ and passing through A.

(c) Find the distance between the points A and B.

(d) Find an equation in general form ($ax+by+cz=d$) for the plane perpendicular to $\overrightarrow{AB}$ and passing through B.

(e) Let C be the point $(-9,16,4)$. Are A, B, and C collinear? (In other words, would the points A, B, and C all be able to fit on a single line?) Show your work.

15. (3 points) Do the points $P(2,-1,1), Q(2,4,0), R(-1,5,2)$ form a right-angle triangle? Justify your answer.

16. (4 points) Find a parametric vector equation of the line describing the intersection of the planes $P_1 : 2x+3y+8z=9$ and $P_2 : x-3y-14z=9$.

17. (6 points) For each set below, determine if it is a subspace of $\mathbb{R}^3$, and justify your answer.

(a) $S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{lcccccc} & 5x & - & 3y & + & z & = 0 \\ \& & 2x & & & + & z & = 0 \end{array} \right\}$

(b) $S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 2x \geq y \right\}$

18. (2 points) Let $H = \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$.

(a) Find a basis for H.

(b) What is dim(H)?

19. Let $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, $\mathbf{a_4}$, $\mathbf{a_5}$, and $\mathbf{a_6}$ be the column vectors of the matrix A below. Knowing that the matrix R results from placing the matrix A in RREF, answer the questions below:

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 0 & -4 \\ -4 & 2 & 6 & -2 & -1 & -1 \\ 2 & 1 & 3 & -9 & 1 & 6 \\ -6 & 3 & 9 & -3 & 0 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 3 & -5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (1 point) Only one of the following sets can be considered a valid basis of Col(A). Which one is it? Circle the correct answer.

$$\{\mathbf{a_1}, \mathbf{a_3}, \mathbf{a_5}\} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

 Question 19 continues on the next page.

(b) (3 points) Find a basis for Nul(A).

(c) (1 point) Is $\mathbf{x} = \begin{bmatrix} 7 \\ 4 \\ 1 \\ 2 \\ 0 \\ 7 \end{bmatrix} \in \text{Nul}(A)$? Show your work.

(d) (2 points) For each of the sets of vectors below, indicate whether they are linearly dependent or linearly independent.

i. $\{\mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}\}$ ii. $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_5}, \mathbf{0}\}$

(e) (2 points) is the vector $\mathbf{a_6}$ in Span $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}\}$? Briefly justify.

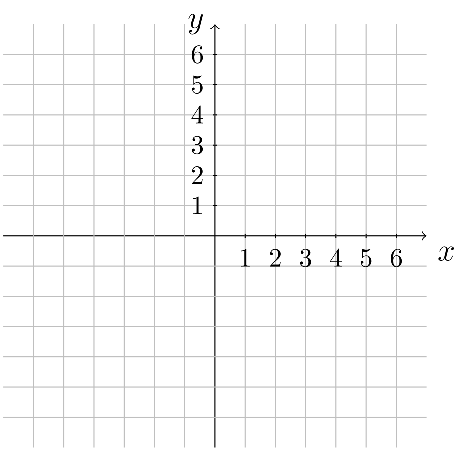
20. (1 point) Let A be a 12×7 matrix for which the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. What is the rank of A?

21. (2 points) Let A be a 12×10 matrix with rank 9.

(a) What is the nullity of A?

(b) What is the nullity of A^T?

22. (1 point) Draw Span $\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$ on the grid below:



23. A certain simple economy has two sectors: goods and services. To produce \$1 of goods requires \$0.40 of goods and \$0.20 of services. To produce \$1 of services requires \$0.10 of goods and \$0.55 of services. There is an external demand for \$6000 of goods and \$13000 of services.

(a) (3 points) Find the production schedule which will exactly meet the external demand.

(b) (1 point) What is the internal consumption of goods and services when the external demand is satisfied?

24. (4 points) Answer true or false. If true, give an explanation. If false give a brief explanation or a counter-example:

 Question 24 continues on the next page.

(a) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ a & b & c \end{bmatrix}$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(b) If A is a 3×5 matrix, then the columns of A might form a linearly independent set.

(c) If AB is an $n \times n$ matrix, A and B must also be $n \times n$ matrices.

(d) If the nullity of a matrix A is zero, that means that no vectors exist in $\mathrm{Nul}(A)$.

25. (2 points) A craftsman makes custom puzzle boxes on demand, and he also makes end tables.

He only builds puzzle boxes when he has orders for them, but he builds end tables without specific orders, knowing that he can always find a buyer for end tables. He currently has 14 puzzle box orders to fill, but only of them 3 of them *must* be filled by the end of the week.

Each puzzle box requires 6 hours to build and requires \$10 worth of materials.

Each end table requires 1 hour to build and requires \$15 worth of materials.

This week, the craftsman has \$180 available to purchase materials, but he cannot spend more than 25 hours building.

The profit earned from the sale of a puzzle box is \$160, and the profit earned from the sale of an end table is \$60. How can the craftsman maximize his profit this week? (You may assume that all end tables built will be sold and that the craftsman is paid for the puzzle boxes once he fills an order.)

State the objective function and list all of the constraints in the form of inequalities. Define what each variable represents. **DO NOT SOLVE THE OPTIMIZATION PROBLEM.**

26. (4 points) Use the graphical method to perform the following optimization:

$$\begin{array}{llrcrcr} \text{MAXIMIZE} & z = & x & + & 4y & & \\ \text{subject to} & & -x & + & 4y & \geq & 0 \\ & & 3x & + & 8y & \leq & 18 \\ & & & & x & \geq & 0 \\ & & & & y & \geq & 0 \end{array}$$

Show your work.

27. (6 points) Use the Simplex Method to verify that, in the situation described below, z has no maximum.

$$\begin{array}{llrcrcr} \text{MAXIMIZE} & z = & -2x_1 & + & 4x_2 & & \\ \text{subject to} & & -10x_1 & + & x_2 & \leq & 7 \\ & & -x_1 & + & x_2 & \leq & 4 \\ & & -2x_1 & + & 3x_2 & \leq & 14 \\ & & & x_1, & x_2 & \geq & 0 \end{array}$$

Demonstrate that z can have very large positive values by finding a basic feasible solution (including slack variables) that would allow z to be 10 020.

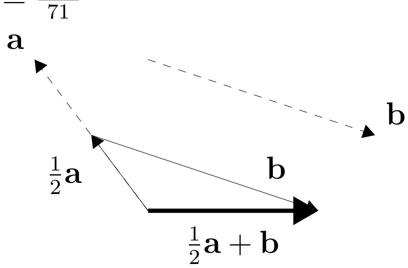
 Question 27 continues on the next page.

Answers

1. $x = 1, y = 0, z = 5$
2. (a) $h \neq 2, k$ can be any real number (b) $h = 2, k \neq \frac{7}{2}$ (c) $h = 2, k = \frac{7}{2}$
3. 2 bats, 1 snake, 16 kangaroos OR 6 bats, 0 snakes, 18 kangaroos
4. (a) undefined (b) $\begin{bmatrix} -11 & 11 & -8 \\ 22 & -22 & 16 \end{bmatrix}$ (c) $\begin{bmatrix} 10 & -14 \\ -21 & 45 \end{bmatrix}$
5. $k = -10$
6. Mistake: $(A+X)^{-1}$ is not equivalent to $A^{-1} + X^{-1}$, Corrected solution:

$$\begin{aligned} ((A+X)B)^{-1} &= C \\ B^{-1}(A+X)^{-1} &= C \\ (A+X)^{-1} &= BC \\ A+X &= (BC)^{-1} \\ A+X &= C^{-1}B^{-1} \\ X &= C^{-1}B^{-1} - A \end{aligned}$$

7. $A^{-1} = \begin{bmatrix} 0 & 5 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{-1}{7} & 0 \end{bmatrix}$
8. (a) -2 (b) -4
9. (a) $-200\,000$ (b) 16
10. (a) $\operatorname{adj}(A) = \begin{bmatrix} 7 & 1 & -3 \\ 4 & 8 & 2 \\ -2 & -4 & 12 \end{bmatrix}$ (b) $\det(A) = 26$ (c) $A^{-1} = \begin{bmatrix} 7/26 & 1/26 & -3/26 \\ 2/13 & 4/13 & 1/13 \\ -1/13 & -2/13 & 6/13 \end{bmatrix}$ (d) $X = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$
11. $z = \frac{-12}{71}$



12.

13. Many solutions possible, like $\left\{ \begin{array}{l} x = s \\ y = t \\ z = -5 + 2s + 3t \end{array} \right\}$
14. (a) $\overrightarrow{AB} = \begin{bmatrix} -4 \\ 6 \\ -4 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ -4 \end{bmatrix}$ (c) $2\sqrt{17}$ units (d) $-2x + 3y - 2z = 12$ (e) No
15. No
16. $\mathbf{x} = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

 Question 27 continues on the next page.

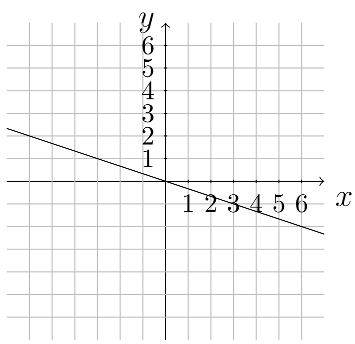
17. (a) $S_1 = \text{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right\}$ and all spans are subspaces, so S_1 is a subspace. (b) S_2 is not a subspace because it is not closed under scalar multiplication. (Counter-example: $\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$ is from S_2, but its scalar multiple $\begin{bmatrix} -3 \\ -5 \\ -9 \end{bmatrix}$ is not from S_2.)

18. (a) $\left\{\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}\right\}$ (b) 2

19. (a) $\{\mathbf{a_1}, \mathbf{a_3}, \mathbf{a_5}\}$ (b) $\left\{\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -7 \\ 1 \end{bmatrix}\right\}$ (c) No (d) i. independent ii. dependent (e) Yes, since $\mathbf{a_6} = -1\mathbf{a_1} + \mathbf{a_2} + 7\mathbf{a_5}$

20. 7

21. (a) 1 (b) 3



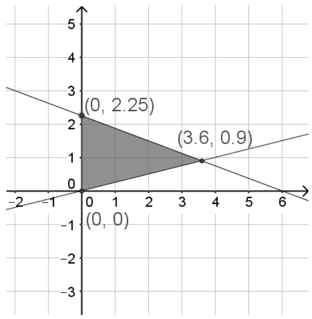
22.

23. (a) \$16 000 in goods and \$36 000 in services (b) \$10 000 in goods and \$23 000 in services

24. (a) TRUE (We cannot obtain three pivots in RREF because the first and last rows of A are identical.) (b) FALSE (There will be at least two free variable columns in the RREF of A.) (c) FALSE (A can be $n \times m$ and B can be $m \times n$.) (d) FALSE ($\mathbf{0}$ is in the null space of any matrix.)

25. Maximize $z = 160x + 60y$ subject to $\left\{\begin{array}{rcrcr} 3 & \leq & x & \leq & 14 \\ 6x & + & y & \leq & 25 \\ 10x & + & 15y & \leq & 180 \\ & & y & \geq & 0 \end{array}\right\}$ if we let x represent the number of puzzle boxes and y represent the number of end tables.

 Question 27 continues on the next page.

26. 

The maximum value of z is 9, obtained by letting $x = 0$ and $y = \frac{9}{4}$.

27. $z = 10\,020$ if $x_1 = 15\,002, x_2 = 10\,006, s_1 = 140\,021, s_2 = 5\,000, s_3 = 0$