

1. (3 points) Solve the following system or show that it is inconsistent.
$$\begin{cases} 3x - 2y + z = 8 \\ -3x + 3y - 2z = -13 \\ -6x + 7y + 4z = 14 \end{cases}$$
2. (4 points) Given $\begin{cases} x + hy = k \\ 2x + 4y = 7 \end{cases}$ For what value(s) of h and k does the system have
- one solution?
 - no solutions?
 - infinitely many solutions?
3. (5 points) David makes plush animals from rags. A bat requires 20 grams of stuffing and 2 rags. A giant snake requires 240 grams of stuffing and 26 rags. A kangaroo requires 80 grams of stuffing and 9 rags. David has 1560 grams of stuffing and 174 rags, and he would like to use them all up. How many bats, snakes and kangaroos should he make? List all realistic possibilities.
4. (4 points) Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ -5 & 1 \end{bmatrix}$. Find, or identify as undefined:
- A^{-1}
 - $(CA)^T$
 - $A^2 + 3BB^{-1}$
5. (3 points) Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & 5 \\ 7 & k \end{bmatrix}$. For what value(s) of k is the product AB symmetric?
6. (3 points) Let A , B , C and X be $n \times n$ matrices. Below, someone was trying to solve for X in the matrix equation $((A + X)B)^{-1} = C$ but made one or more mistakes! Identify the mistake(s) and then solve for X correctly. (You may assume that any necessary matrices are invertible.)

Equation to solve for X : $((A + X)B)^{-1} = C$

Incorrect solution:

$$\begin{aligned} ((A + X)B)^{-1} &= C \\ B^{-1}(A + X)^{-1} &= C \\ (A + X)^{-1} &= BC \\ A^{-1} + X^{-1} &= BC \\ X^{-1} &= BC - A^{-1} \\ X &= (BC - A^{-1})^{-1} \end{aligned}$$

Corrected solution:

7. (3 points) $A = \begin{bmatrix} 0 & 4 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 3 & 0 \end{bmatrix}$. Find A^{-1} or show that it does not exist.

8. (3 points) Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, evaluate the following determinants:

(a) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

(b) $\begin{vmatrix} a & b & c \\ 2d+3a & 2e+3b & 2f+3c \\ -g & -h & -i \end{vmatrix}$

9. (3 points) If A is a 5×5 matrix with $\det(A) = -2$,

(a) Find the determinant of $10A^T$.

(b) Find the determinant of $\text{adj}(A)$.

10. Given: $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 4 \\ -9 \end{bmatrix}$,

(a) (3 points) Find $\text{adj}(A)$

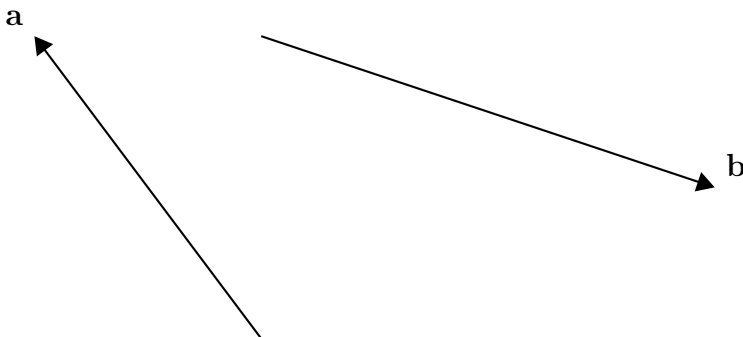
(b) (2 points) Compute $\det(A)$

(c) (1 point) Find A^{-1}

(d) (2 points) Use A^{-1} to solve the system $AX = B$

11. (4 points) Use Cramer's Rule to solve for z in the system:
$$\begin{cases} x + 5y + 5z = 0 \\ 4x + 7y + 6z = 2 \\ 2x + 4y + 9z = 0 \end{cases}$$

12. (1 point) Given \mathbf{a} and \mathbf{b} below, draw (and clearly label) the vector $\frac{1}{2}\mathbf{a} + \mathbf{b}$.



13. (3 points) Find equations for the plane $2x + 3y - z = 5$ in parametric form.

14. (5 points) Let $A = (3, -2, 5)$ and $B = (-1, 4, 1)$

- (a) Find \overrightarrow{AB} .
- (b) Find a parametric vector equation for the line parallel to \overrightarrow{AB} and passing through A .
- (c) Find the distance between the points A and B .
- (d) Find an equation in general form ($ax + by + cz = d$) for the plane perpendicular to \overrightarrow{AB} and passing through B .
- (e) Let C be the point $(-9, 16, 4)$. Are A , B , and C collinear? (In other words, would the points A , B , and C all be able to fit on a single line?) Show your work.

15. (4 points) Find a parametric vector equation of the line describing the intersection of the planes $P_1 : 2x + 3y + 8z = 9$ and $P_2 : x - 3y - 14z = 9$.

16. (6 points) For each set below, determine if it is a subspace of \mathbb{R}^3 , and justify your answer.

$$(a) S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid \begin{array}{l} 5x - 3y + z = 0 \\ 2x \quad \quad + z = 0 \end{array} \right\}$$

$$(b) S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 2x \geq y \right\}$$

17. (2 points) Let $H = \text{Span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$.

- (a) Find a basis for H .
- (b) What is $\dim(H)$?

18. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$, and \mathbf{a}_6 be the column vectors of the matrix A below. Knowing that the matrix R results from placing the matrix A in RREF, answer the questions below:

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 0 & -4 \\ -4 & 2 & 6 & -2 & -1 & -1 \\ 2 & 1 & 3 & -9 & 1 & 6 \\ -6 & 3 & 9 & -3 & 0 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 3 & -5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- (a) (1 point) Only one of the following sets can be considered a valid basis of $\text{Col}(A)$. Which one is it? Circle the correct answer.

$$\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- (b) (3 points) Find a basis for $\text{Nul}(A)$.

(c) (1 point) Is $\mathbf{x} = \begin{bmatrix} 7 \\ 4 \\ 1 \\ 2 \\ 0 \\ 7 \end{bmatrix} \in \text{Nul}(A)$? Show your work.

(d) (2 points) For each of the sets of vectors below, indicate whether they are linearly dependent or linearly independent.

i. $\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$

ii. $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5, \mathbf{0}\}$

(e) (2 points) is the vector \mathbf{a}_6 in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$? Briefly justify.

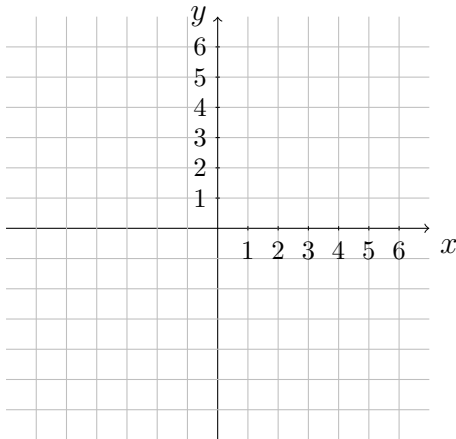
19. (1 point) Let A be a 12×7 matrix for which the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. What is the rank of A ?

20. (2 points) Let A be a 12×10 matrix with rank 9.

(a) What is the nullity of A ?

(b) What is the nullity of A^T ?

21. (1 point) Draw $\text{Span}\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right\}$ on the grid below:



22. (3 points) A certain simple economy has two sectors: goods and services. To produce \$1 of goods requires \$0.40 of goods and \$0.20 of services. To produce \$1 of services requires \$0.10 of goods and \$0.55 of services. There is an external demand for \$6000 of goods and \$13000 of services. Find the production schedule which will exactly meet the external demand.

23. (3 points) Answer true or false. If true, give an explanation. If false give a brief explanation or a counter-example:

(a) If A is a 3×5 matrix, then the columns of A might form a linearly independent set.

(b) If AB is an $n \times n$ matrix, A and B must also be $n \times n$ matrices.

(c) If the nullity of a matrix A is zero, that means that no vectors exist in $\text{Nul}(A)$.

24. (2 points) A craftsman makes custom puzzle boxes on demand, and he also makes end tables.

He only builds puzzle boxes when he has orders for them, but he builds end tables without specific orders, knowing that he can always find a buyer for end tables. He currently has 14 puzzle box orders to fill, but only of them 3 of them *must* be filled by the end of the week.

Each puzzle box requires 6 hours to build and requires \$10 worth of materials.

Each end table requires 1 hour to build and requires \$15 worth of materials.

This week, the craftsman has \$180 available to purchase materials, but he cannot spend more than 25 hours building.

The profit earned from the sale of a puzzle box is \$160, and the profit earned from the sale of an end table is \$60. How can the craftsman maximize his profit this week? (You may assume that all end tables built will be sold and that the craftsman is paid for the puzzle boxes once he fills an order.)

State the objective function and list all of the constraints in the form of inequalities. Define what each variable represents. **DO NOT SOLVE THE OPTIMIZATION PROBLEM.**

25. (4 points) Use the graphical method to perform the following optimization:

$$\begin{array}{l} \text{MAXIMIZE } z = x + 4y \\ \text{subject to} \quad -x + 4y \geq 0 \\ \quad \quad \quad 3x + 8y \leq 18 \\ \quad \quad \quad x \geq 0 \\ \quad \quad \quad y \geq 0 \end{array}$$

Show your work.

26. In a specific population, it has been verified that there is a $1/2$ probability that an individual with blue eyes have a child with blue eyes. Meanwhile, if an individual has brown eyes, then the probability of them having a child with blue eyes is only $1/8$.

- (1 point) Find a transition matrix P associated with this situation.
- (2 points) Jean is a member of the population who has blue eyes. What is the probability that his first grandchild will also have blue eyes?
- (3 points) In the distant future, what is the long-term probability that a child is born with blue eyes? (You may, for the sake of this question, assume that we are considering a child born in infinitely many generations.)

27. (5 points) A secret agent in the midwest, you must meet your informant at one of the following public locations:

Al's Diner
Big Church
Cemetery
Clock Tower

High School
Library
Mill Road
O'Tool's Pub

Old Mine
Park Gates
Red Bridge
Theatre

Town Square

Water Park

Yacht Club

You were asked to memorize the encryption matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ during your training, and the encrypted message **XQSWQJAVFJ** is left for you at the front desk of your hotel. Use your knowledge of the Hill 2-cipher to determine where you are supposed to meet your informant.

You may use process of elimination to solve this problem, but show your work, including the decryption matrix that you've used. You may find the following table of multiplicative inverses mod (26) helpful:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

Answers

1. $x = 1, y = 0, z = 5$

2. (a) $h \neq 2, k$ can be any real number (b) $h = 2, k \neq \frac{7}{2}$ (c) $h = 2, k = \frac{7}{2}$

3. 2 bats, 1 snake, 16 kangaroos OR 6 bats, 0 snakes, 18 kangaroos

4. (a) undefined (b) $\begin{bmatrix} -11 & 11 & -8 \\ 22 & -22 & 16 \end{bmatrix}$ (c) $\begin{bmatrix} 10 & -14 \\ -21 & 45 \end{bmatrix}$

5. $k = -10$

6. Mistake: $(A + X)^{-1}$ is not equivalent to $A^{-1} + X^{-1}$, Corrected solution:

$$((A + X)B)^{-1} = C$$

$$B^{-1}(A + X)^{-1} = C$$

$$(A + X)^{-1} = BC$$

$$A + X = (BC)^{-1}$$

$$A + X = C^{-1}B^{-1}$$

$$X = C^{-1}B^{-1} - A$$

7. $A^{-1} = \begin{bmatrix} 0 & 5 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{-1}{7} & 0 \end{bmatrix}$

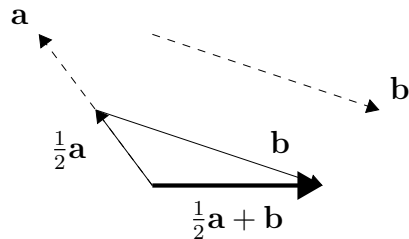
8. (a) -2 (b) -4

9. (a) $-200\,000$ (b) 16

10. (a) $\text{adj}(A) = \begin{bmatrix} 7 & 1 & -3 \\ 4 & 8 & 2 \\ -2 & -4 & 12 \end{bmatrix}$ (b) $\det(A) = 26$ (c) $A^{-1} = \begin{bmatrix} 7/26 & 1/26 & -3/26 \\ 2/13 & 4/13 & 1/13 \\ -1/13 & -2/13 & 6/13 \end{bmatrix}$ (d) $X =$

$$\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

11. $z = \frac{-12}{71}$



12.

13. Many solutions possible, like $\left\{ \begin{array}{l} x = s \\ y = t \\ z = -5 + 2s + 3t \end{array} \right\}$

14. (a) $\overrightarrow{AB} = \begin{bmatrix} -4 \\ 6 \\ -4 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} -4 \\ 6 \\ -4 \end{bmatrix}$ (c) $2\sqrt{17}$ units (d) $-2x + 3y - 2z = 12$ (e) No

15. $\mathbf{x} = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

16. (a) $S_1 = \text{Span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ and all spans are subspaces, so S_1 is a subspace. (b) S_2 is not a subspace

because it is not closed under scalar multiplication. (Counter-example: $\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$ is from S_2 , but its scalar

multiple $\begin{bmatrix} -3 \\ -5 \\ -9 \end{bmatrix}$ is not from S_2 .)

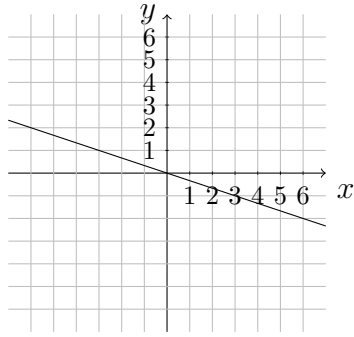
17. (a) $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ (b) 2

18. (a) $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ (b) $\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -7 \\ 1 \end{bmatrix} \right\}$ (c) No (d) i. independent ii. dependent (e) Yes, since

$$\mathbf{a}_6 = -1\mathbf{a}_1 + \mathbf{a}_2 + 7\mathbf{a}_5$$

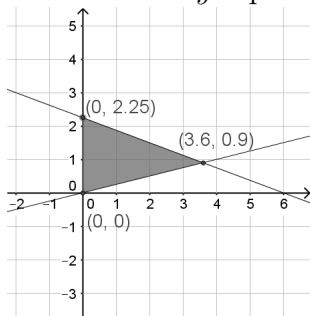
19. 7

20. (a) 1 (b) 3



- 21.
22. \$16 000 in goods and \$36 000 in services 23. (a) FALSE (There will be at least two free variable columns in the RREF of A .) (b) FALSE (A can be $n \times m$ and B can be $m \times n$.) (c) FALSE ($\mathbf{0}$ is in the null space of any matrix.)

24. Maximize $z = 160x + 60y$ subject to $\left\{ \begin{array}{l} 3 \leq x \leq 14 \\ 6x + y \leq 25 \\ 10x + 15y \leq 180 \\ y \geq 0 \end{array} \right\}$ if we let x represent the number of puzzle boxes and y represent the number of end tables.



25. The maximum value of z is 9, obtained by letting $x = 0$ and $y = \frac{9}{4}$.
26. (a) $P = \begin{bmatrix} 1/2 & 1/8 \\ 1/2 & 7/8 \end{bmatrix}$ (b) $5/16$ (c) $1/5$
27. O'TOOL'S PUB (using $A^{-1} = \begin{bmatrix} 8 & 11 \\ 11 & 4 \end{bmatrix}$)