1. (5 points) Given $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 6 \\ 2 & 2 & 5 & -1 & 18 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

(a) Express the general solution of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

(b) Given that $\begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{d}$, express the general solution to $A\mathbf{x} = \mathbf{d}$ in parametric vector form.

2. (5 points) **Use the matrix method** to balance the chemical equation:

$$\underline{\qquad} KClO_3 \to \underline{\qquad} KCl + \underline{\qquad}\ O_2$$

3. (5 points) Let $A = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 3 & 1 \\ -2 & -3 & 0 \end{bmatrix}$.

(a) Find the inverse of A.

(b) What is $(A^T)^{-1}$?

4. (3 points) Compute the determinant of $\begin{bmatrix} 2a+3b & abd - b^2c \\ 2c+3d & ad^2 - bcd \end{bmatrix}$ given that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 8$. (You may find it helpful to factor the entries wherever possible.)

5. (5 points) You are given the following matrix. $A = \begin{bmatrix} 3 & -1 & 1 \\ 9 & 2 & 5 \\ -6 & 22 & 10 \end{bmatrix}$

(a) Write an LU decomposition for A.

(b) Write the matrix L as a product of elementary matrices.

6. (9 points) Let

$$A = \begin{bmatrix} 1 & a & 2 & 1 & e \\ 2 & b & 3 & 2 & f \\ 3 & c & -1 & -1 & g \\ -4 & d & 4 & 1 & h \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}.$$

Assuming that R is the reduced row echelon form of the matrix A, answer the following questions.

(a) What are the vectors $\mathbf{u}$ and $\mathbf{v}$?

(b) Find a basis for $\mathrm{Col}(A)$.

(c) How many vectors are in $\mathrm{Col}(A)$?

(d) Find a basis for $\mathrm{Nul}(A)$.

 Question 6 continues on the next page.

(e) For what value(s) of k is $\begin{bmatrix} -25 \\ 13 \\ 5 \\ k \end{bmatrix}$ in $\mathrm{Nul}(A^T)$?

(f) TRUE or FALSE: $\mathrm{Nul}(A^T)$ is a line.

7. (3 points) Assume that all matrices given below are $n \times n$ and invertible, solve for the matrix X in

$$B(X + A)^{-1} = C$$

8. (7 points) Let A be a 4×4 matrix with $\det(A) = -3$, and let I be the 4×4 identity matrix. Furthermore, assume that $A = LU$ where L is unit lower triangular and U is upper triangular. Calculate:

(a) $\det(L)$

(b) $\det(U)$

(c) $\det(2(A^T)^3 A^{-1})$

(d) $\det(LA + A)$

9. (2 points) Suppose that A is an $n \times n$ matrix. Show that if $\mathrm{Nul}(A)$ has dimension zero, then $\mathrm{Nul}(A^2)$ must also have dimension zero.

10. (2 points) Give an example of a non-invertible 2×2 matrix A, for which $\det(A + I) = 0$.

11. (5 points) Let $H = \left\{ A \in M_{2\times 2} \;:\; A \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(a) Find a specific nonzero matrix that is in H.

(b) Given that H is a subspace, find a basis for it.

12. (6 points) Let $V = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$ and $W = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

(a) Show that if A is any matrix in V then A^2 will be in W.

(b) TRUE or FALSE: V is a 2-dimensional subspace of W.

(c) TRUE or FALSE: W is a 3-dimensional subspace of V.

13. (8 points) Let $V = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \;:\; w = z \;\text{ and }\; xy = z^2 \right\}$.

(a) Is $\mathbf{0}$ in V?

(b) Find a nonzero vector in V.

 Question 13 continues on the next page.

(c) Is V closed under scalar multiplication? Justify your answer.

(d) Is V closed under vector addition? Justify your answer.

(e) Is V a subspace of $\mathbb{R}^4$?

14. (6 points) Let $\mathcal{T} = \triangle ABC$ denote the triangle whose vertices are the points $A(-2,6,8)$, $B(-3,9,12)$, and $C(0,6,9)$.

(a) Is the inner angle at the vertex B in $\mathcal{T}$ acute (between 0 and $\frac{\pi}{2}$ radians) or obtuse (between $\frac{\pi}{2}$ and π radians). Explain your answer.

(b) Find an equation of the form $ax + by + cz = d$ for the plane through the point $P(1,0,-1)$ that is parallel to the plane containing the triangle $\mathcal{T}$.

15. (5 points) Let $\mathcal{L}$ denote the line given by the parametric vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, and let P denote the point $(7,10,3)$. Find the distance from $\mathcal{L}$ to P.

16. (5 points) Consider the line $\mathcal{L}$ in $\mathbb{R}^3$ given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and the plane $\mathcal{P}$ given by $x - 2y + 2z = -8$.

(a) Find the points on the line $\mathcal{L}$ that are 1 unit away from the plane $\mathcal{P}$.

(b) Find the point where $\mathcal{L}$ and $\mathcal{P}$ intersect.

17. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates vectors clockwise around the origin by θ, then reflects through the x axis, then rotates again by θ clockwise, and then reflects through the y axis. If $T(\mathbf{x}) = A\mathbf{x}$, find A. (Your final answer should not depend on the angle θ.)

18. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -27 \\ 13 \end{bmatrix},\ T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -27 \\ 13 \end{bmatrix}$$

(a) Based on the given conditions is T one-to-one? Explain your answer.

(b) Express $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Is the transformation T linear? Justify.

19. (2 points) Let $T : U \to V$ be a linear transformation. Show that if $T(\mathbf{u}_1) = T(\mathbf{u}_2)$ then $2\mathbf{u}_1 - 2\mathbf{u}_2$ is in the kernel of T.

20. (7 points) Fill in the blanks with the word **must**, **might**, or **cannot**, as appropriate.

(a) The non pivot columns of a matrix A __________ form a linearly dependent set.

Question 20 continues on the next page.

(b) If A is an 5×8 matrix and $\operatorname{rank}(A)=5$ then the linear transformation $T(\mathbf{x})=A\mathbf{x}$ ________ be onto and ________ be one-to-one.

(c) If $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a linearly independent set in $\operatorname{Span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$, then $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ ________ be a linearly independent set.

(d) The columns of an elementary matrix ________ form a linearly independent set.

(e) If $\operatorname{Col}(A)=\operatorname{Col}(A^T)$ for a $n\times n$ matrix A, then A ________ be a symmetric matrix.

(f) Given an $n\times n$ matrix A. If the system $A\mathbf{x}=\mathbf{b}$ is inconsistent for some $\mathbf{b}\in\mathbb{R}^n$, then the system $A\mathbf{x}=\mathbf{0}$ ________ have non-trivial solutions.

Answers

1. (a) $\mathbf{x}=\begin{bmatrix}-10\\0\\3\\0\\0\end{bmatrix}+r\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}3\\0\\-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-14\\0\\2\\0\\1\end{bmatrix}$

(b) $\mathbf{x}=\begin{bmatrix}4\\1\\2\\1\\-1\end{bmatrix}+r\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}3\\0\\-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-14\\0\\2\\0\\1\end{bmatrix}$

2. 2 $KClO_3 \to$ 2 KCl +3 O_3

3. (a) $A^{-1}=\begin{bmatrix}3&0&-2\\-2&0&1\\6&1&-3\end{bmatrix}$ (b) $(A^T)^{-1}=\begin{bmatrix}3&-2&6\\0&0&1\\-2&1&-3\end{bmatrix}$

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5. (a) $A=\begin{bmatrix}1&0&0\\3&1&0\\-2&4&1\end{bmatrix}\begin{bmatrix}3&-1&1\\0&5&2\\0&0&4\end{bmatrix}$ (b) $L=\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&4&1\end{bmatrix}$

6. (a) $\mathbf{u}=\begin{bmatrix}3\\6\\9\\-12\end{bmatrix},\mathbf{v}=\begin{bmatrix}6\\11\\3\\-2\end{bmatrix}$ (b) $\left\{\begin{bmatrix}1\\2\\3\\-4\end{bmatrix},\begin{bmatrix}2\\3\\-1\\4\end{bmatrix},\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}\right\}$ (c) Infinitely many

(d) $\left\{\begin{bmatrix}-3\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\-1\\-2\\1\end{bmatrix}\right\}$ (e) $k=4$ (f) TRUE

7. $X=C^{-1}B-A$

8. (a) 1 (b) -3 (c) 144 (d) $\det(L+I)\det(A)=-24$

 Question 20 continues on the next page.

9. $\dim(\text{Nul}(A)) = 0 \Rightarrow A$ is invertible and has $\det(A) \neq 0 \Rightarrow \det(A^2) = [\det(A)]^2 \neq 0 \Rightarrow A^2$ is also invertible and has $\dim(\text{Nul}(A^2)) = 0$

10. $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ (many answers possible)

11. (a) $\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ (many answers possible) (b) $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ (many answers possible)

12. (a) $A^2 = \left(k_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right)^2 = {k_1}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2(k_1+k_2) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + {k_2}^2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
$\Rightarrow A^2 \in W$ (b) TRUE (c) FALSE (not a subset of V)

13. (a) Yes. (b) $\begin{bmatrix} 4 \\ -2 \\ -8 \\ 4 \end{bmatrix}$ (many answers possible) (c) Yes. If $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in V$, then $w = z$ and $xy = z^2$ are true. So $k \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} kw \\ kx \\ ky \\ kz \end{bmatrix} \in V$ since $k(w) = k(z)$ and $(kx)(ky) = k^2(xy) = k^2(z^2) = (kz)^2$

(d) No. Counter-example: $\begin{bmatrix} 4 \\ -2 \\ -8 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ -8 \\ -4 \end{bmatrix} \in V$, but their sum $\begin{bmatrix} 0 \\ -4 \\ -16 \\ 0 \end{bmatrix} \notin V$. (e) No.

14. (a) Since $\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\| \|\overrightarrow{BC}\|} > 0$, the inner angle at the vertex B must be acute. (b) $-x - 3y + 2z = -3$

15. $\frac{3\sqrt{6}}{2}$ units

16. (a) $(\frac{3}{2}, \frac{23}{4}, \frac{-1}{2})$ and $(\frac{-3}{2}, \frac{17}{4}, \frac{5}{2})$ (b) $(0, 5, 1)$

17. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

18. (a) No. $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$ and $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$ yield the same result. (b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
(c) No. $T\left(\frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \neq \frac{5}{2} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + \frac{-1}{2} T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$.

19. $T(2\mathbf{u_1} - 2\mathbf{u_2}) = 2T(\mathbf{u_1}) - 2T(\mathbf{u_2}) = 2T(\mathbf{u_1}) - 2T(\mathbf{u_1}) = \mathbf{0} \Rightarrow 2\mathbf{u_1} - 2\mathbf{u_2} \in \ker(T)$

20. (a) MIGHT (b) MUST, CANNOT (c) MUST (d) MUST (e) MIGHT
(f) MUST