1. Solve each of the following systems or show that it is inconsistent.

(a) $\begin{cases} 3x - y + z = 6 \\ 4x - y = 5 \\ -2x - y + 3z = 5 \end{cases}$

(b) $\begin{cases} 4x_1 + x_2 + 2x_3 + 4x_4 = 4 \\ 3x_1 - x_2 + 3x_3 + x_4 = 3 \\ 2x_1 + 4x_2 - 2x_3 + 6x_4 = 4 \end{cases}$

2. Given the system $\begin{cases} x + y = 3 \\ x + 2y = k+4 \\ x + (k+1)y = 3k+3 \end{cases}$

Find the value(s) of k, if any, such that

(a) The system is consistent.

(b) The system is inconsistent.

3. Miss Muffit's Bakery is well known for three types of muffins; Loaded Muffins, Fruity Muffins, and Nutty Muffins. The main ingredients aside from flour, eggs, sugar, and water are fruit and nuts. Each batch of Loaded Muffins uses 200 grams of fruit and 200 grams of nuts. Each batch of Fruity Muffins uses 400 grams of fruit and no nuts. Each batch of Nutty Muffins uses 100 grams of fruit and 500 grams of nuts. The bakery has 3000 grams of fruit and 1000 grams of nuts on hand and wants to use up all of the ingredients. If only complete batches are allowed, list all realistic possibilities for the number of batches of each muffin type that can be made.

4. Find the intersection of the planes $5x - 2y + 3z = 6$ and $2x - y + z = 1$.
Express this intersection in vector form.

5. Given $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$, $D = \begin{bmatrix} a & 2 & 3 \\ b^2 - 7 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$,

and the **inverse** of A is $A^{-1} = \begin{bmatrix} -2 & 4 \\ 8 & 3 \end{bmatrix}$;

Find the following, or indicate that it is not possible, as appropriate:

(a) $(AB)^{-1}$

(b) $C^T B^2$

(c) The value(s) of a and b, if any, so that the matrix D is symmetric.

6. A simple economy has two industries: Hot and Cold. The production of 1\$ of Hot requires 40¢ of Hot and 20¢ of Cold. The production of 1\$ of Cold requires 5¢ of Hot and 65¢ of Cold. There is an external demand of \$600 of Hot and \$200 of Cold.

(a) Which, if either, of the two industries are profitable? Justify.

(b) What dollar amount should each industry produce?

(c) What is the internal consumption?

7. Let A, B, and C be 3×3 matrices, and let $\det(A) = -3, \det(B) = 5$, and C be non-invertible. Find the following, or state that there is not enough information, as appropriate:

 Question 7 continues on the next page.

(a) $\det(A^{-1}BA)$

(b) $\det(AC - BC)$

(c) $\det(2B^{-1})$

(d) Rank of C

(e) Column space of A

8. Given $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & -1 & -2 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix}$ and the linear system $\begin{cases} x + 2y + z + w = 0 \\ 2x + y + 3z = 0 \\ x - y - 2z + w = 0 \\ 2y + z + 3w = -2 \end{cases}$

(a) Find the determinant of A.

(b) Use Cramer's Rule to solve for **z only** in the linear system.

9. If $A = \begin{bmatrix} -2 & 2 & -4 \\ 0 & -1 & 3 \\ 5 & 7 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$:

(a) Find $\operatorname{adj}(A)$.

(b) Compute $\det(A)$.

(c) Find A^{-1} using $\operatorname{adj}(A)$.

(d) Solve $AX = B$ using A^{-1}.

10. Given the points $A(3, 1, 2), B(0, 1, 6)$ and the line $\mathcal{L}_1 : \begin{cases} x = 2 - 8t \\ y = 1 + t \\ z = -6t \end{cases}$

(a) Find the magnitude of $\overrightarrow{AB}$.

(b) Find a unit vector in the direction of $\overrightarrow{AB}$.

(c) Find parametric equations for the line $\mathcal{L}_2$ passing through A and B.

(d) Compare the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Are they parallel, orthogonal, or neither? Justify.

11. *Utensibles* produces forks, spoons and knives at two factories. When working at maximum efficiency, Factory A produces 2000 forks, 4000 spoons, and 2000 knives per day, while Factory B working at maximum efficiency produces 6000 forks, 6000 spoons, and 3000 knives per day. The company receives an order for 3000 forks, 4000 spoons, and 2000 knives. Is there a combination of days of production at the two factories that would allow Utensibles to fill this order exactly while operating at maximum efficiency?

12. Given $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ k \end{bmatrix}$;

(a) Describe the span of $\mathbf{u}$ and $\mathbf{v}$.
If it is a line, use vector form. If it is a plane, use the general form ($ax + by + cz = d$).

 Question 12 continues on the next page.

(b) Is $\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$ in the span of $\mathbf{u}$ and $\mathbf{v}$? Show some work.

(c) Find all value(s) of k, if any, so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

(d) Find all value(s) of k, if any, so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbb{R}^3$.

13. Let $A = \begin{bmatrix} 1 & 2 & 4 & a & 2 \\ 5 & 1 & 11 & b & 5 \\ 4 & 1 & 9 & c & 5 \end{bmatrix}$, which reduces to $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, and let $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}, \mathbf{u_5}$ represent the columns of A.

(a) For each of the following sets, determine whether it is linearly dependent or linearly independent. Justify.
(i) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$
(ii) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_5}\}$
(iii) $\{\mathbf{u_3}, \mathbf{u_5}\}$

(b) Is $\mathbf{u_1}$ in the span of $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_5}\}$? Justify.

(c) Which of the following are a valid basis for the column space of A? (Circle your answer.)

(i) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_4}\}$ (ii) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (iii) both (iv) neither

(d) What are the values of $a, b,$ and c in the vector $\mathbf{u_4}$?

(e) Find a basis for the Null space of A.

(f) Given that $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \mathbf{b}$, find the general solution to $A\mathbf{x} = \mathbf{b}$.

14. Suppose that A and B are both 4×5 matrices.

(a) If the rank of A is 3, what is the nullity of A^T?

(b) If the nullity of B^T is 4, what is the nullity of B?.

15. Find a basis for the subspace $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| x + 7y = 0 \right\}$.

16. Let the set $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| xz = 0 \right\}$.

(a) Is $(0, 0, 0)$ in S?

(b) Give three vectors in S that are not scalar multiples.

(c) Is S a subspace of $\mathbb{R}^3$? Justify.

17. Solve the linear program by graphing the feasibility region:

$$\text{Maximize } P = 3x + 3y \text{ subject to } \begin{cases} -2x + y \leq 4 \\ x - 2y \leq 6 \\ x + y \leq 6 \\ x, \; y \geq 0 \end{cases}$$

18. An airline has a plane with 190 seats, for which they sell three types of tickets; first class for \$310 per ticket, executive class for \$270 per ticket, and economy class for \$140 per ticket. The airline wants to sell **at least 10 more** executive class tickets than first class tickets. The airline's cost is \$150 for a first class ticket, \$150 for an executive ticket, and \$50 for an economy ticket. The cost of each flight cannot exceed \$22500.
Additionally, the airline must reserve 10 cubic feet of storage space for each first class passenger and 5 cubic feet of storage space for each economy class passenger, while the executive class passengers have no luggage to check and therefore require no storage space. The plane has a maximum storage space available of 1350 cubic feet.

You have been hired to help them determine how many tickets of each type should be sold to maximize **revenue**.

Set up the linear programming problem as follows:

(a) Define all of the variables used.

(b) State the objective and identify the objective function (in terms of the variables from part (a)).

(c) State all of the constraints (in terms of the variables from part (a)).

DO NOT SOLVE.

Answers

1. (a) $x = 1, y = -1, z = 2$ (b) inconsistent
2. (a) $k = 0, 2$ (b) $k \neq 0, 2$
3. 5 Batches of Loaded Muffins, 5 Fruity, 0 Nutty **OR** 0 Loaded, 7 Fruity, 2 Nutty
4. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
5. (a) $\begin{bmatrix} \frac{-32}{5} & \frac{7}{5} \\ \frac{18}{5} & \frac{2}{5} \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 149 \\ 5 & 20 \\ 10 & 15 \end{bmatrix}$ (c) a can be any real number, $b = \pm 3$
6. (a) Both industries are profitable since it costs less than \$1 to produce \$1 worth of Hot or of Cold. (b) \$1100 Hot and \$1200 Cold (c) \$500 Hot and \$1000 Cold
7. (a) 5 (b) $\det(A - B)\det(C) = 0$ (c) $\frac{8}{5}$ (d) not enough information (e) all of $\mathbb{R}^3$
8. (a) 30 (b) $\frac{-2}{5}$

 Question 18 continues on the next page.

9. (a) $\text{adj}(A) = \begin{bmatrix} -22 & -30 & 2 \\ 15 & 18 & 6 \\ 5 & 24 & 2 \end{bmatrix}$ (b) 54 (c) $A^{-1} = \begin{bmatrix} \frac{-11}{27} & \frac{-5}{9} & \frac{1}{27} \\ \frac{5}{18} & \frac{1}{3} & \frac{1}{9} \\ \frac{5}{54} & \frac{4}{9} & \frac{1}{27} \end{bmatrix}$ (d) $X = \begin{bmatrix} \frac{-62}{27} \\ \frac{29}{18} \\ \frac{65}{54} \end{bmatrix}$

10. (a) 5 units (b) $\begin{bmatrix} \frac{-3}{5} \\ 0 \\ \frac{-2}{5} \end{bmatrix}$ (c) $\{x = 3 - 3s, y = 1, z = 2 + 4s\}$ (d) orthogonal

11. Yes, it is possible: $\frac{1}{2}$ production of Factory A $+\frac{1}{3}$ production of Factory B = production of Factory C.

12. (a) the plane $-x + y - z = 0$ (b) Yes. (c) $k = 2$ (d) $k \neq 2$

13. (a) (i) dependent; (ii) independent; (iii) independent (b) Yes, since $\text{Span}\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_5}\} = \mathbb{R}^3$.

(c) iii (both) (d) $a = 1, b = 0, c = 1$ (e) $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(f) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

14. (a) 1 (b) 5

15. $\left\{ \begin{bmatrix} -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (many answers possible)

16. (a) Yes. (b) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 18 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}$ (many answers possible) (c) No, S is not closed under addition. (Counter-example: $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in S$, $\begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} \in S$, but their sum is $\begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix} \notin S$.)

17. Vertices of the basic feasible region: $(0,0), (0,3), (\frac{2}{3}, \frac{16}{3}), (6,0)$. The maximum value of P is 18, occurring both when $x = \frac{2}{3}, y = \frac{16}{3}$ AND when $x = 6, y = 0$.

18. (a) x_1: the number of first class tickets, x_2: the number of executive class tickets, x_3: the number of economy class tickets (b) Maximize $R = 310x_1 + 270x_2 + 140x_3$

(c) $$\left\{ \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \leq & 190 \\ 150x_1 & + & 15x_2 & + & 5x_3 & \leq & 22500 \\ 10x_1 & & & + & 5x_3 & \leq & 1350 \\ & & x_2 & - & x_1 & \geq & 10 \\ & & x_1, & x_2, & x_3 & \geq & 0 \end{array} \right\}$$