1. (6 points) Solve each of the following systems or show that it is inconsistent.

(a) $\begin{cases} 3x - y + z = 6 \\ 4x - y = 5 \\ -2x - y + 3z = 5 \end{cases}$

(b) $\begin{cases} 4x_1 + x_2 + 2x_3 + 4x_4 = 4 \\ 3x_1 - x_2 + 3x_3 + x_4 = 3 \\ 2x_1 + 4x_2 - 2x_3 + 6x_4 = 4 \end{cases}$

2. (4 points) Given the system $\begin{cases} x + y = 3 \\ x + 2y = k+4 \\ x + (k+1)y = 3k+3 \end{cases}$

Find the value(s) of k, if any, such that

(a) The system is consistent.

(b) The system is inconsistent.

3. (5 points) Miss Muffit's Bakery is well known for three types of muffins; Loaded Muffins, Fruity Muffins, and Nutty Muffins. The main ingredients aside from flour, eggs, sugar, and water are fruit and nuts. Each batch of Loaded Muffins uses 200 grams of fruit and 200 grams of nuts. Each batch of Fruity Muffins uses 400 grams of fruit and no nuts. Each batch of Nutty Muffins uses 100 grams of fruit and 500 grams of nuts. The bakery has 3000 grams of fruit and 1000 grams of nuts on hand and wants to use up all of the ingredients. If only complete batches are allowed, list all realistic possibilities for the number of batches of each muffin type that can be made.

4. (3 points) Find the intersection of the planes $5x - 2y + 3z = 6$ and $2x - y + z = 1$. Express this intersection in vector form.

5. (4 points) Given $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 & 4 \\ 5 & 2 & -3 \end{bmatrix}, D = \begin{bmatrix} a & 2 & 3 \\ b^2-7 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$,

and the **inverse** of A is $A^{-1} = \begin{bmatrix} -2 & 4 \\ 8 & 3 \end{bmatrix}$;

Find the following, or indicate that it is not possible, as appropriate:

(a) $(AB)^{-1}$

(b) $C^T B^2$

(c) The value(s) of a and b, if any, so that the matrix D is symmetric.

6. (5 points) A simple economy has two industries: Hot and Cold. The production of 1\$ of Hot requires 40¢ of Hot and 20¢ of Cold. The production of 1\$ of Cold requires 5¢ of Hot and 65¢ of Cold. There is an external demand of \$600 of Hot and \$200 of Cold.

(a) Which, if either, of the two industries are profitable? Justify.

(b) What dollar amount should each industry produce?

(c) What is the internal consumption?

7. (5 points) Let A, B, and C be 3×3 matrices, and let $\det(A) = -3, \det(B) = 5$, and C be non-invertible. Find the following, or state that there is not enough information, as appropriate:

 Question 7 continues on the next page.

(a) $\det(A^{-1}BA)$

(b) $\det(AC - BC)$

(c) $\det(2B^{-1})$

(d) Rank of C

(e) Column space of A

8. (6 points) Given $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & -1 & -2 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix}$ and the linear system $\begin{cases} x + 2y + z + w = 0 \\ 2x + y + 3z = 0 \\ x - y - 2z + w = 0 \\ 2y + z + 3w = -2 \end{cases}$

(a) Find the determinant of A.

(b) Use Cramer's Rule to solve for **z only** in the linear system.

9. (6 points) If $A = \begin{bmatrix} -2 & 2 & -4 \\ 0 & -1 & 3 \\ 5 & 7 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$:

(a) Find adj(A).

(b) Compute $\det(A)$.

(c) Find A^{-1} using adj(A).

(d) Solve $AX = B$ using A^{-1}.

10. (4 points) Given the points $A(3, 1, 2), B(0, 1, 6)$ and the line $\mathcal{L}_1 : \begin{cases} x = 2 - 8t \\ y = 1 + t \\ z = -6t \end{cases}$

(a) Find the magnitude of $\overrightarrow{AB}$.

(b) Find a unit vector in the direction of $\overrightarrow{AB}$.

(c) Find parametric equations for the line $\mathcal{L}_2$ passing through A and B.

(d) Compare the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Are they parallel, orthogonal, or neither? Justify.

11. (4 points) *Utensibles* produces forks, spoons and knives at two factories. When working at maximum efficiency, Factory A produces 2000 forks, 4000 spoons, and 2000 knives per day, while Factory B working at maximum efficiency produces 6000 forks, 6000 spoons, and 3000 knives per day. The company receives an order for 3000 forks, 4000 spoons, and 2000 knives. Is there a combination of days of production at the two factories that would allow Utensibles to fill this order exactly while operating at maximum efficiency?

12. (6 points) Given $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ k \end{bmatrix}$;

(a) Describe the span of $\mathbf{u}$ and $\mathbf{v}$.
If it is a line, use vector form. If it is a plane, use the general form $(ax + by + cz = d)$.

 Question 12 continues on the next page.

(b) Is $\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$ in the span of $\mathbf{u}$ and $\mathbf{v}$? Show some work.

(c) Find all value(s) of k, if any, so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

(d) Find all value(s) of k, if any, so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbb{R}^3$.

13. Let $A = \begin{bmatrix} 1 & 2 & 4 & a & 2 \\ 5 & 1 & 11 & b & 5 \\ 4 & 1 & 9 & c & 5 \end{bmatrix}$, which reduces to $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, and let $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}, \mathbf{u_5}$ represent the columns of A.

(a) (3 points) For each of the following sets, determine whether it is linearly dependent or linearly independent. Justify.
(i) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$
(ii) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_5}\}$
(iii) $\{\mathbf{u_3}, \mathbf{u_5}\}$

(b) (1 point) Is $\mathbf{u_4}$ in the span of $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_5}\}$? Justify.

(c) (1 point) Which of the following are a valid basis for the column space of A? (Circle your answer.)

(i) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_4}\}$ (ii) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (iii) both (iv) neither

(d) (2 points) What are the values of a, b, and c in the vector $\mathbf{u_4}$?

(e) (3 points) Find a basis for the Null space of A.

(f) (1 point) Given that $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \mathbf{b}$, find the general solution to $A\mathbf{x} = \mathbf{b}$.

14. (2 points) Suppose that A and B are both 4×5 matrices.

(a) If the rank of A is 3, what is the nullity of A^T?

(b) If the nullity of B^T is 4, what is the nullity of B?.

15. (3 points) Find a basis for the subspace $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| x + 7y = 0 \right\}$.

16. (5 points) Let the set $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| xz = 0 \right\}$.

(a) Is $(0, 0, 0)$ in S?

(b) Give three vectors in S that are not scalar multiples.

(c) Is S a subspace of $\mathbb{R}^3$? Justify.

17. (6 points) Solve the linear program by graphing the feasibility region:

$$\text{Maximize } P = 3x + 3y \text{ subject to } \begin{cases} -2x + y \leq 4 \\ x - 2y \leq 6 \\ x + y \leq 6 \\ x, \; y \geq 0 \end{cases}$$

18. A local grocery store relies on a number of different producers to supply them with vegetables every week. If the store receives their vegetables from a local grower on any given week, there is a 40% probability that the following week's shipment will also come from a local grower. However, if the vegetables on any given week are supplied by a grower outside of the area, there is a 70% probability that the following week's shipment will also come from a grower outside of the area.

(a) (1 point) Find a transition matrix P associated with the situation described.

(b) (3 points) If this week's shipment of vegetables came from a grower outside of the area, what is the probability that the grocery store will receive a shipment from a local grower two weeks afterwards?

(c) (3 points) Find a steady-state vector $\mathbf{q}$ associated with the transition matrix P found in part (a).

(d) (1 point) What is the long-term probability (infinitely many weeks in the future) that the grocery store will receive a shipment from a local grower?

19. (7 points) Before you left home this morning, your father left you the following ciphered text:

JYWAEZTE

Decode the message if you know that it consists of a Hill 2-cipher that uses an encoding matrix $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$.

You may find the following table of multiplicative inverses mod (26) helpful:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

Answers

1. (a) $x = 1, y = -1, z = 2$ (b) inconsistent
2. (a) $k = 0, 2$ (b) $k \neq 0, 2$
3. 5 Batches of Loaded Muffins, 5 Fruity, 0 Nutty **OR** 0 Loaded, 7 Fruity, 2 Nutty
4. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
5. (a) $\begin{bmatrix} \frac{-32}{5} & \frac{7}{5} \\ \frac{18}{5} & \frac{2}{5} \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 149 \\ 5 & 20 \\ 10 & 15 \end{bmatrix}$ (c) a can be any real number, $b = \pm 3$

 Question 19 continues on the next page.

6. (a) Both industries are profitable since it costs less than \$1 to produce \$1 worth of Hot or of Cold. (b) \$1100 Hot and \$1200 Cold (c) \$500 Hot and \$1000 Cold

7. (a) 5 (b) $\det(A-B)\det(C)=0$ (c) $\frac{8}{5}$ (d) not enough information (e) all of $\mathbb{R}^3$

8. (a) 30 (b) $\frac{-2}{5}$

9. (a) $\operatorname{adj}(A)=\begin{bmatrix}-22 & -30 & 2\\ 15 & 18 & 6\\ 5 & 24 & 2\end{bmatrix}$ (b) 54 (c) $A^{-1}=\begin{bmatrix}\frac{-11}{27} & \frac{-5}{9} & \frac{1}{27}\\ \frac{5}{18} & \frac{1}{3} & \frac{1}{9}\\ \frac{5}{54} & \frac{4}{9} & \frac{1}{27}\end{bmatrix}$ (d) $X=\begin{bmatrix}\frac{-62}{27}\\ \frac{29}{18}\\ \frac{65}{54}\end{bmatrix}$

10. (a) 5 units (b) $\begin{bmatrix}\frac{-3}{5}\\ 0\\ \frac{4}{5}\end{bmatrix}$ (c) $\{x=3-3s, y=1, z=2+4s\}$ (d) orthogonal

11. Yes, it is possible: $\frac{1}{2}$ production of Factory A $+\frac{1}{3}$ production of Factory B = production of Factory C.

12. (a) the plane $-x+y-z=0$ (b) Yes. (c) $k=2$ (d) $k\neq 2$

13. (a) (i) dependent; (ii) independent; (iii) independent (b) Yes, since $\operatorname{Span}\{\mathbf{u_1},\mathbf{u_2},\mathbf{u_5}\}=\mathbb{R}^3$.

(c) iii (both) (d) $a=1, b=0, c=1$ (e) $\left\{\begin{bmatrix}-2\\-1\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\0\\-1\\1\end{bmatrix}\right\}$

(f) $\mathbf{x}=\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}+s\begin{bmatrix}-2\\-1\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\0\\-1\\1\end{bmatrix}$

14. (a) 1 (b) 5

15. $\left\{\begin{bmatrix}-7\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ (many answers possible)

16. (a) Yes. (b) $\begin{bmatrix}0\\1\\2\end{bmatrix},\begin{bmatrix}0\\-4\\18\end{bmatrix},\begin{bmatrix}-6\\3\\0\end{bmatrix}$ (many answers possible) (c) No, S is not closed under addition. (Counter-example: $\begin{bmatrix}0\\1\\2\end{bmatrix}\in S$, $\begin{bmatrix}-6\\3\\0\end{bmatrix}\in S$, but their sum is $\begin{bmatrix}-6\\4\\2\end{bmatrix}\notin S$.)

17. Vertices of the basic feasible region: $(0,0),(0,4),(\frac{2}{3},\frac{16}{3}),(6,0)$. The maximum value of P is 18, occurring both when $x=\frac{2}{3}, y=\frac{16}{3}$ AND when $x=6, y=0$.

18. (a) $P=\begin{bmatrix}0.4 & 0.3\\ 0.6 & 0.7\end{bmatrix}$ (b) 33% (c) $\mathbf{q}=\begin{bmatrix}1/3\\ 2/3\end{bmatrix}$ (d) 1/3

19. GOOD LUCK (using $A^{-1}=\begin{bmatrix}18 & 17\\ 25 & 1\end{bmatrix}$)