1. Evaluate each of the following expressions.

a. $7-[-3^2-(-2)^2]^2+\frac{2}{5}\left(\frac{50}{4}\right)$ b. $\frac{9}{2}\div\left(\frac{5}{6}+\frac{11}{3}\right)\times\frac{8}{3}$ c. $3\div\frac{5(-9)}{3^0-|-2|}$

2. Expand each of the following expressions and collect like terms.

a. $-[2x-(7x-2)]^2+\frac{2}{3}(9-6x)$ b. $3(x+4)(x-\frac{1}{2})-(2x+1)(2x-1)$

c. $(3x-2)^3$

3. Simplify each of the following expressions. Be sure to leave no square under a square root sign. In Parts a and b, assume that the letters x, y and z represent positive numbers.

a. $xyz\sqrt{125x^7y^4z^2}$ b. $\dfrac{-z\sqrt{84x^4y^5z^8}}{x\sqrt{7x^4y^9z^3}}$

c. $2\sqrt{45}-3\sqrt{18}+\sqrt{72}-\sqrt{20}$ d. $(3\sqrt{2}+\sqrt{6})(\sqrt{6}-4\sqrt{2})$ e. $(2\sqrt{3}+3\sqrt{5})^2$

4. Solve each equation for x.

a. $3(7-2x+x^2)=14+3x^2-8(x-1)$ b. $\frac{2}{3}x-\frac{1}{5}=\frac{1}{2}\left(\frac{5}{6}-\frac{3}{5}x\right)$

5. Simplify each expression and express the result without using negative exponents.

a. $\left(\dfrac{24a^5b^{-6}}{48a^2b^{-7}}\right)^{-1}$ b. $\dfrac{(3x^2y^2)^{-2}}{(2x^{-1}y^0)^3}\cdot\dfrac{3x^2}{4y}$

6. The 2016 Smart Toaster is on sale at \$85 after a 32% discount. What was the original price of the toaster?

7. Factorize each expression completely.

a. x^4-13x^2+36 b. $16s^4-2st^3$

8. Solve each equation for x by factorizing.

a. $4x^2-28x=120$ b. $15x^2-4x=3x+4$ c. $2x^3-9x^2=8x-36$

9. Rationalize the denominator and simplify the result. Be sure to leave no square under a square root sign.

a. $\dfrac{15}{\sqrt{5}}$ b. $\dfrac{4\sqrt{2}}{\sqrt{6}+3\sqrt{2}}$

10. Solve the equation $\sqrt{4-12x}-6=2x$ for x.

11. Solve the equation $x^2-6=3x$ for x *by completing the square*.

12. Solve the equation $3x^2+4x+5=0$ for x *by using the Quadratic Formula*.

13. Solve the equation $\frac{1}{4}(x-7)^2-30=-5$ for x *by taking square roots*.

14. The hypotenuse of a right-angled triangle is ten inches long. One of the legs of the triangle is two inches shorter than the other leg. Find the lengths of the legs of the triangle.

15. Solve the following system of linear equations *by using substitution*.

$$\begin{aligned} 3x+2y&=7\\ -x+y&=1 \end{aligned}$$

16. Solve the following system of linear equations *by elimination*.

$$\begin{aligned} 3x-5y&=3\\ 4x-7y&=1 \end{aligned}$$

17. Let ℓ be the line defined by $4x=5y-10$.

a. Find the axis intercepts of ℓ. b. Find the slope of ℓ.

c. Sketch the graph of ℓ.

d. Is ℓ parallel, perpendicular, or neither parallel nor perpendicular, to the line with equation $y=\frac{4}{5}x+3$?

18. You are given the points $A(4,-1)$ and $B(3,6)$.

a. Find the distance between A and B.

b. Find the midpoint of the line segment AB.

c. Find an equation of the line which:

i. passes through A and B;

ii. contains B and is perpendicular to the line defined by $y=3x+2$;

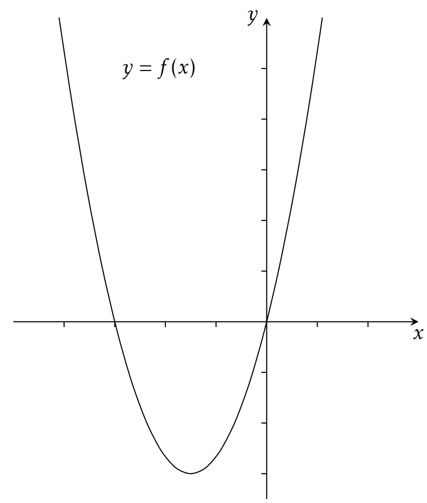
iii. contains A and is parallel to the line with equation $x=-1$.

19. Find the point of intersection of the lines $3x+y=4$ and $5x+6y=-2$.

20. Let $f(x)=x^2+6x+4$. Find: a. $f(2)$; b. $f\left(\frac{1}{3}\right)$; c. $f(a)$; d. $f(a+h)$;

e. the value(s) of x for which $f(x)=-4$.

21. Below is a sketch of the graph of a function f, with unit lengths marked along the coordinate axes. (Assume that the curve continues as it appears it should.)



a. Find the domain and range of f.

b. Find the axis intercepts of the curve.

c. Determine the interval(s) on which f is positive, and the interval(s) on which f is negative.

d. Determine the interval(s) on which f is increasing and the interval(s) on which f is decreasing.

e. Find all extreme values of f.

f. Determine $f(x)$, given that it is a quadratic polynomial.

22. Solve each equation for x. Where possible, express your answer without using logarithms.

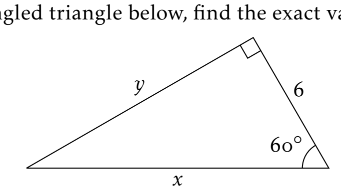
a. $3^{2x-3}=5^{3-x}$ b. $2(e^{2x/3}-4)=7$

c. $\log_2(x^2-2x)=3$ d. $2^x+\dfrac{24}{2^x}=11$

23. Given that $\sin\vartheta=\frac{7}{9}$ and ϑ is an acute angle of a right-angled triangle, find the exact values of $\cos\vartheta$, $\tan\vartheta$, $\cot\vartheta$, $\csc\vartheta$ and $\sec\vartheta$.

24. Given that $\cot\vartheta=\sqrt{3}$, find the acute angle ϑ.

25. In the right-angled triangle below, find the exact values of x and y.



Solutions

1. a. First evaluate the second and third terms. This gives

$$[-3^2-(-2)^2]^2 = (-9-4)^2 = (-13)^2 = 169$$

and

$$\tfrac{2}{5}\left(\tfrac{50}{4}\right) = \tfrac{2}{5}\cdot\tfrac{25}{2} = 5.$$

Now, combining these results gives

$$7-[-3^2-(-2)^2]^2+\tfrac{2}{5}\left(\tfrac{50}{4}\right) = 7-169+5 = -157.$$

b. Since

$$\tfrac{5}{6}+\tfrac{11}{3} = \tfrac{5}{6}+\tfrac{22}{6} = \tfrac{27}{6} = \tfrac{9}{2},$$

it follows that

$$\tfrac{9}{2}\div\left(\tfrac{5}{6}+\tfrac{11}{3}\right)\times\tfrac{8}{3} = \tfrac{9}{2}\div\tfrac{9}{2}\times\tfrac{8}{3} = 1\times\tfrac{8}{3} = \tfrac{8}{3}.$$

c. The numerator and denominator of the divisor are

$$5(-9) = -45 \qquad\text{and}\qquad 3^0-|-2| = 1-2 = -1,$$

so

$$3\div\frac{5(-9)}{3^0-|-2|} = 3\div 45 = \tfrac{3}{45} = \tfrac{1}{15}.$$

2. a. Since

$$[2x-(7x-2)]^2 = (2-5x)^2 = 4-20x+25x^2,$$

and

$$\tfrac{2}{3}(9-6x) = 6-4x,$$

it follows that

$$\begin{aligned}-[2x-(7x-2)]^2+\tfrac{2}{3}(9-6x) &= -4+20x-25x^2+6-4x\\ &= 2+16x-25x^2.\end{aligned}$$

b. Expanding the terms individually gives

$$3(x+4)(x-\tfrac{1}{2}) = 3(x^2+\tfrac{7}{2}x-2) = 3x^2+\tfrac{21}{2}x-6$$

and

$$(2x+1)(2x-1) = 4x^2-1.$$

Thus,

$$\begin{aligned}3(x+4)(x-\tfrac{1}{2})-(2x+1)(2x-1) &= 3x^2+\tfrac{21}{2}x-6-4x^2+1\\ &= -x^2+\tfrac{21}{2}x-5.\end{aligned}$$

c. The binomial theorem gives

$$\begin{aligned}(3x-2)^3 &= (3x)^3+3(3x)^2\cdot(-2)+3(3x)(-2)^2+(-2)^3\\ &= 27x^3-54x^2+36x-8.\end{aligned}$$

3. a. First observe that

$$125 = 5^2\cdot 5, \qquad x^7 = (x^3)^2\cdot x \qquad\text{and}\qquad y^4 = (y^2)^2.$$

Hence (since it is given that z is positive),

$$xyz\sqrt{125x^7y^4z^2} = (xyz)(5x^3y^2z)\sqrt{5x} = 5x^4y^3z^2\sqrt{5x}.$$

b. Since $84 = 12\cdot 7$ and $12 = 2^2\cdot 3$, the numerical coefficient is $-\sqrt{12} = -2\sqrt{3}$. Next, the exponent of x is $(4-4)/2-1 = -1$, the exponent of y is $(5-9)/2 = -2$ and the exponent of z is $1+(8-3)/2 = 1+\frac{5}{2} = \frac{7}{2}$. Therefore,

$$\frac{-z\sqrt{84x^4y^5z^8}}{x\sqrt{7x^4y^9z^3}} = -2\sqrt{3}\,x^{-1}y^{-2}z^{7/2} = -\frac{2z^3\sqrt{3z}}{xy^2}.$$

(Either form is acceptable, as negative/rational exponents are not forbidden.)

c. Since $45 = 3^2\cdot 5$, $18 = 3^2\cdot 2$, $72 = 6^2\cdot 2$ and $20 = 2^2\cdot 5$, it follows that

$$\begin{aligned}2\sqrt{45}-3\sqrt{18}+\sqrt{72}-\sqrt{20} &= 2\cdot 3\sqrt{5}-3\cdot 3\sqrt{2}+6\sqrt{2}-2\sqrt{5}\\ &= 4\sqrt{5}-3\sqrt{2}.\end{aligned}$$

d. Since $\sqrt{2}\sqrt{6} = 2\sqrt{3}$, expanding gives

$$(3\sqrt{2}+\sqrt{6})(\sqrt{6}-4\sqrt{2}) = 6\sqrt{3}-24+6-8\sqrt{3} = -18-2\sqrt{3}.$$

e. Since $(2\sqrt{3})^2 = 12$ and $(3\sqrt{5})^2 = 45$, squaring the binomial gives

$$(2\sqrt{3}+3\sqrt{5})^2 = 12+2(2\sqrt{3})(3\sqrt{5})+45 = 57+12\sqrt{15}.$$

4. a. The equation

$$3(7-2x+x^2) = 14+3x^2-8(x-1)$$

is equivalent to

$$21-6x+3x^2 = 22-8x+3x^2, \qquad\text{or}\qquad 2x = 1.$$

So the solution to the equation in question is $\frac{1}{2}$.

b. Since $\frac{1}{2}\left(\frac{5}{6}-\frac{3}{5}x\right) = \frac{5}{12}-\frac{3}{10}x$, the equation

$$\tfrac{2}{3}x-\tfrac{1}{5} = \tfrac{1}{2}\left(\tfrac{5}{6}-\tfrac{3}{5}x\right),$$

is equivalent to

$$\left(\tfrac{2}{3}+\tfrac{3}{10}\right)x = \tfrac{5}{12}+\tfrac{1}{5}, \qquad\text{or}\qquad \tfrac{29}{30}x = \tfrac{37}{60}.$$

So the solution to the equation in question is $\frac{30}{29}\cdot\frac{37}{60} = \frac{37}{58}$.

5. a. Since $48 = 24\cdot 2$, the numerical coefficient is $\left(\frac{1}{2}\right)^{-1} = 2$. The exponent of a is $-(5-2)- = -3$ and the exponent of b is $-(-6-(-7)) = -1$. Hence,

$$\left(\frac{24a^5b^{-6}}{48a^2b^{-7}}\right)^{-1} = 2a^{-3}b^{-1} = \frac{2}{a^3b}.$$

b. The exponent of 3 is $-2+1 = -1$ and the exponent of 2 is $-3-2 = -5$, so the numerical coefficient is $3^{-1}2^{-5} = \frac{1}{3}\cdot\frac{1}{32} = \frac{1}{96}$. The exponent of x is $-2(2)-3(-1)+2 = 1$ and the exponent of y is $-2(2)-3(0)-1 = -5$. Therefore,

$$\frac{(3x^2y^2)^{-2}}{(2x^{-1}y^0)^3}\cdot\frac{3x^2}{4y} = \tfrac{1}{96}xy^{-5} = \frac{x}{96y^5}.$$

6. If the \$85 sale price of the toaster is the original price discounted by 32%, then \$85 is 68%, or $\frac{17}{25}$, of the original price. Therefore, the original price is $\$85\cdot\frac{25}{17} = \$125$.

7. a. Factorizing by inspection, and then observing that each factor is a difference of squares, gives

$$x^4-13x^2+36 = (x^2-9)(x^2-4) = (x+3)(x-3)(x+2)(x-2).$$

b. After a common factor is extracted, the result is a difference of cubes:

$$16s^4-2st^3 = 2s(8s^3-t^3) = 2s(2s-t)(4s^2+2st+t^2).$$

Since $4s^2+2s+1 = \frac{1}{4}(16s^2+8s+4) = \frac{1}{4}((4s+1)^2+3)$, the expression is factorized completely.

8. a. Dividing by 4 and collecting all terms on the left side of the equation gives

$$x^2-7x-30 = 0, \qquad\text{or}\qquad (x+3)(x-10) = 0.$$

So the solutions of the equation are -3 and 10.

b. Collecting all terms on the left side of the equation gives

$$15x^2-7x-4 = 0, \qquad\text{or}\qquad (3x+1)(5x-4) = 0.$$

So the solutions of the equation are $-\frac{1}{3}$ and $\frac{4}{5}$.

c. Since $2x^3-9x^2 = x^2(2x-9)$ and $8x-36 = 4(2x-9)$, it follows that

$$2x^3-9x^2-(8x-36) = (x^2-4)(2x-9) = (x+2)(x-2)(2x-9).$$

The solutions of the equation are the zeros of this polynomial: -2, 2 and $\frac{9}{2}$.

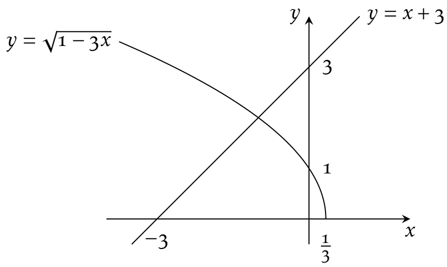
9. a. Since $15 = 3\cdot 5 = 3\cdot(\sqrt{5})^2$, it follows that

$$\frac{15}{\sqrt{5}} = 3\sqrt{5}.$$

b. If $a = 3$ and $b = \sqrt{3}$, then $a^2-b^2 = 6$, so $1/(a+b) = \frac{1}{6}(a-b)$. Thus, cancelling the common factor $\sqrt{2}$ gives

$$\frac{4\sqrt{2}}{\sqrt{6}+3\sqrt{2}} = \frac{4}{3+\sqrt{3}} = \tfrac{4}{6}(3-\sqrt{3}) = 2-\tfrac{2}{3}\sqrt{3}.$$

10. The equation $\sqrt{4-12x}-6=2x$ is equivalent to $\sqrt{1-3x}=x+3$. By inspection, this equation is true if $x=-1$. From the graphs of $y=\sqrt{1-3x}$ and $y=x+3$, (displayed below), it is clear that the equation has exactly one solution. So -1 is *the* solution to the equation.



11. The equation $x^2-6=3x$ is equivalent to $4x^2-12x=24$, *i.e.*, $(2x-3)^2=33$, or $2x-3=\pm\sqrt{33}$. So the solutions of the equation are $\frac{3}{2}\pm\frac{1}{2}\sqrt{33}$.

12. The discriminant of $3x^2+4x+5=0$ is $4^2-4\cdot3\cdot5=-44$, which is negative, so the equation has no solution.

13. The equation $\frac{1}{4}(x-7)^2-30=-5$ is equivalent to $(x-7)^2=100$, or $x-7=\pm10$. So the solutions of the equation are -3 and 17.

14. Since $y^2+(y+2)^2$ increases with y when y is positive, there is one, and only one, value of y for which $y^2+(y+2)^2=10^2$, so the triangle must be a "3–4–5" right-angled triangle whose sides are measured in units of two inches. Therefore, the legs are six inches long and eight inches long.

15. The equation $-x+y=1$ is equivalent to $y=x+1$. Replacing y by $x+1$ in the equation $3x+2y=7$ gives $3x+2(x+1)=7$, or $5x=5$. So the solution of the system is $(1,2)$.

16. By adding $-\frac{4}{3}$ of the first equation to the second, the system

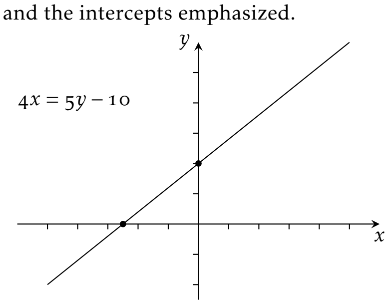
$$\begin{aligned}3x-5y&=3\\4x-7y&=1,\end{aligned}\qquad\text{is equivalent to}\qquad\begin{aligned}3x-5y&=3\\-\tfrac{1}{3}y&=-3.\end{aligned}$$

The last equation is equivalent to $y=9$, which gives $3x=45+3=48$, or $x=16$. So the solution of the system is $(16,9)$.

17. A standard form of the equation of ℓ is $4x-5y=-10$.
a. The equation $4x-5\cdot0=-10$ is equivalent to $x=-\frac{5}{2}$, and the equation $4\cdot0-5y=-10$ is equivalent to $y=2$. So the axis intercepts of ℓ are $(-\frac{5}{2},0)$ and $(0,2)$.

b. The slope of ℓ is $(-4)/(-5)=\frac{4}{5}$.

c. The graph of ℓ is sketched below, with unit lengths marked along the coordinate axes and the intercepts emphasized.



d. The line $y=\frac{4}{5}x+3$ has the same slope, $\frac{4}{5}$, as ℓ, so it is parallel to ℓ.

18. a. The distance between $A(4,-1)$ and $B(3,6)$ is $\sqrt{(-1)^2+7^2}=5\sqrt{2}$.

b. As $\frac{1}{2}(4+3)=\frac{7}{2}$ and $\frac{1}{2}(-1+6)=\frac{5}{2}$, the midpoint of segment AB is $\left(\frac{7}{2},\frac{5}{2}\right)$.

c. i. The translation from $A(4,-1)$ to $B(3,6)$ is $(-1,7)$, and $7\cdot4+1\cdot(-1)=27$, so the line containing A and B is defined by $7x+y=27$.

ii. Since $1\cdot3+3\cdot6=21$, the line which is perpendicular to the line $3x-y=-2$ and contains $B(3,6)$ is defined by $x+3y=21$.

iii. The vertical line which passes through $A(4,-1)$ is defined by $x=4$.

19. A point (x,y) lies on both of the given lines if, and only if,

$$3x+y=4\qquad\text{and}\qquad5x+6y=-2.$$

Adding -6 times the first equation to the second equation yields $-13x=-26$, or $x=2$, and then $y=4-3\cdot2=-2$. So the lines intersect at the point $(2,-2)$.

20. If $f(x)=x^2+6x+4$, then:
a. $f(2)=2^2+6\cdot2+4=20$;
b. $f\left(\frac{1}{3}\right)=\left(\frac{1}{3}\right)^2+6\left(\frac{1}{3}\right)+4=\frac{1}{9}+6=\frac{55}{9}$;
c. $f(a)=a^2+6a+4$;
d. $f(a+h)=(a+h)^2+6(a+h)+4=a^2+6a+4+2ah+6h+h^2$;
e. $f(x)=-4$ gives $x^2+6x+8=0$, or $(x+2)(x+4)=0$; so $x=-2$ or $x=-4$.

21. a. The domain of f is $\mathbb{R}$ and the range of f is $[-3,\infty)$.

b. The axis intercepts of the curve are $(-3,0)$ and the origin.

c. The function of f is positive on the intervals $(-\infty,-3)$ and $(,0,\infty)$, and is negative on the interval $(-3,0)$.

d. The function f is decreasing on the interval $(-\infty,-\frac{3}{2}]$ and is increasing on the interval $[-\frac{3}{2},\infty)$.

e. The function f has no maximum value. The minimum value of f is -3, which occurs at the number $-\frac{3}{2}$ in its domain.

f. If $f(x)$ is a quadratic polynomial, there is a non-zero real number α such that $f(x)=\alpha x(x+3)$. Since $f\left(-\frac{3}{2}\right)=-3$, it follows that $\alpha\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right)=-3$, or $\alpha=\frac{4}{3}$. Therefore, $f(x)=\frac{4}{3}x(x+3)$.

22. a. The equation $3^{2x-3}=5^{3-x}$ is equivalent to $(2x-3)\log3=(3-x)\log5$, *i.e.*, $x(2\log3+\log5)=3(\log3+\log5)$, or $x\log45=3\log15$, whose solution is $(3\log15)/(\log45)$.

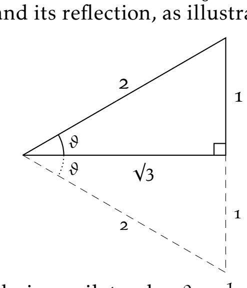
b. The equation $2(e^{2x/3}-4)=7$ equivalent to $e^{2x/3}=\frac{15}{2}$, or $\frac{2}{3}x=\log\frac{15}{2}$, whose solution is $\frac{3}{2}\log\frac{15}{2}$. (*Note*: We use grown-up notation: log is the natural logarithm.)

c. The equation $\log_2(x^2-2x)=3$ is equivalent to $x<0$ or $x>2$, and $x^2-2x=2^3$, *i.e.*, $x^2-2x-8=0$, or $(x+2)(x-4)=0$. So the solutions of the equation are -2 and 4.

d. The equation $2^x+\dfrac{24}{2^x}=11$ is equivalent to $(2^x)^2-11\cdot2^x+24=0$, or $(2^x-3)(2^x-8)=0$. So the solutions of the equation are $\log_2 3$ and 3.

23. If $\sin\vartheta=\frac{7}{9}$, then $(\cos\vartheta)^2=1-\left(\frac{7}{9}\right)^2=\frac{32}{81}$. Since ϑ is acute, $\cos\vartheta=\frac{4}{9}\sqrt{2}$ and $\tan\vartheta=\frac{7}{8}\sqrt{2}$. Also, $\cot\vartheta=\frac{4}{7}\sqrt{2}$, $\csc\vartheta=\frac{9}{7}$ and $\sec\vartheta=\frac{9}{8}\sqrt{2}$.

24. If $\cot\vartheta=\sqrt{3}$ and ϑ is acute, then $\vartheta=\frac{1}{6}\pi$. In more detail, there is a right-angled triangle, and its reflection, as illustrated below.



Since the outer triangle is equilateral, $2\vartheta=\frac{1}{3}\pi$ (one-third of a straight angle); therefore, $\vartheta=\frac{1}{6}\pi$.

25. The right-angled triangle in this question is similar to the one in the preceding question, which implies that $x=2\cdot6=12$ and $y=6\sqrt{3}$.