1. (8 points) Given the following coefficient matrix A and vector $\mathbf{b}$:

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 \\ -1 & -1 & -3 & -3 \\ -2 & -1 & -4 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -6 \\ 6 \\ 7 \\ -5 \end{bmatrix}$$

(a) Find the general solution to $A\mathbf{x} = \mathbf{b}$

(b) Find the specific solution such that $x_1 = x_2$ and $x_3 = x_4$.

(c) Which columns of A (if any) are in the solution set of $A\mathbf{x} = \mathbf{b}$?

(d) Which columns of A (if any) are in the null space of A?

(e) Find a basis for the row space of A.

2. (6 points) Let A and B be 4×4 matrices with $\det A = 3$ and $\det B = -2$. Find the following or indicate that there is not enough information, as necessary:

(a) $\det((2A)^{-1})$

(b) $\det(B^{-1}A^T B)$

(c) $\det(B + B^{-1})$

3. (7 points) Let $A = \begin{bmatrix} 2 & -2 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 6 & 2 \end{bmatrix}$.

(a) Find $\det A$.

(b) How many solutions does the homogeneous system of linear equations $A\mathbf{x} = 0$ have?

4. (2 points) If A is a skew-symmetric $n \times n$ matrix, i.e., $A^T = -A$, show that when n is odd, $\det A = 0$.

5. (6 points) Consider the quadratic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ that passes through the point $(2, -1)$, and that has a tangent line with slope 2 at the point $(1, -6)$.

(a) Find the initial augmented matrix that would allow us to solve for the coefficients of the polynomial $p(x)$. **Do not row reduce the matrix.**

(b) Use Cramer's rule to solve for a_0 only.

6. (3 points) Given that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three linearly independent vectors in $\mathbb{R}^n$. For which value(s) of k will the vectors $\mathbf{u} + 2\mathbf{v}$, $\mathbf{v} + 3\mathbf{w}$ and $k\mathbf{u} + \mathbf{w}$ be linearly dependent?

7. (8 points) (a) Consider the block matrix $A = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$ where B and D are invertible. Find a formula (as a block matrix) for A^{-1}.

(b) Use an appropriate partitioning to find the inverse of $A = \begin{bmatrix} -3 & 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}$.

8. (7 points) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 9 & 1 \\ 6 & -8 & k \end{bmatrix}$.

(a) Find an LU decomposition of A.

(b) Using the LU decomposition from part (a), what is the determinant of A?

(c) For what k is A not invertible?

(d) Write the matrix L as a product of elementary matrices.

9. (6 points) (a) Let V be the set of all 2×2 upper triangular matrices. What is the dimension of V?

(b) Write the matrix $\begin{bmatrix} 19 & 20 \\ 0 & -3 \end{bmatrix}$ as a linear combination of the matrices $\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$.

(c) Does the set $\left\{ \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 19 & 20 \\ 0 & -3 \end{bmatrix} \right\}$ span the subspace of all 2×2 upper triangular matrices? Justify your answer.

10. (3 points) In $\mathbb{R}^3$ let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Let $W = \{\mathbf{x} \in \mathbb{R}^3 \ : \mathbf{u} \cdot \mathbf{x} = 0\}$. Given that W is a subspaces of $\mathbb{R}^3$, find a basis for W.

11. (5 points) (a) Find a standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ are mapped onto the vectors $T(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T(\mathbf{v}_2) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(b) Use your answer in part (a) to find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$.

12. (6 points) Suppose that a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ follows the calculation $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a - 2b \\ b^2 - a \end{bmatrix}$.

(a) Evaluate $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right)$.

(b) Explain why the results in part (a) imply that T is not a linear transformation.

(c) Find a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

13. (11 points) Given the points $A(5, 2, 0)$, $B(7, 0, -2)$, $C(2, 1, 1)$ and $D(4, 3, 4)$.

(a) Find a vector of length equal to 2 units which is parallel to the vector $\overrightarrow{AB}$.

(b) Find an equation for the line containing the points A and B.

(c) Find the distance between the point D and the line found in part (b).

(d) Find the closest point on the line found in part (b) to the point D.

(e) Find the area of the triangle with vertices A, B and C.

14. (4 points) Let V be the set of 2×2 matrices that are not invertible.

(a) Is V closed under scalar multiplication? Justify your answer. (No credit will be given without justification.)

(b) Is V closed under addition? Justify your answer. (No credit will be given without justification.)

15. (4 points) Let A be a 2×2 matrix such that $S(\mathbf{x}) = A\mathbf{x}$ is a reflection.
Let B be a 2×2 matrix such that $T(\mathbf{x}) = B\mathbf{x}$ is a rotation.
Complete each of the following sentences with MUST, MIGHT, or CANNOT.

A^2 ____________ equal A
A^{-1} ____________ equal A
B^3 ____________ equal B
$\det(A^2)$ ____________ equal $\det(B^2)$

16. (8 points) Given the lines $\mathcal{L}_1$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\mathcal{L}_2$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find the point of intersection between the lines.

(b) Determine the cosine of the acute angle formed by the lines.

(c) Find an equation of the form $ax + by + cz = d$ for the plane containing the two lines.

(d) Find the x-intercept of the plane from part (c). (In other words, at what point do the plane and the x-axis meet?)

17. (3 points) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$ such that $\mathbf{u} + 2\mathbf{v}$ is orthogonal to $\mathbf{u} - 2\mathbf{v}$, and $\|\mathbf{u}\| = 1$. Find $\|\mathbf{v}\|$.

18. (3 points) Given the planes $\mathcal{P}_1$: $x_1 + 2x_2 + x_3 = 4$ and $\mathcal{P}_2$: $2x_1 + 5x_2 + 3x_3 = 1$, find an equation for the line parallel to both $\mathcal{P}_1$ and $\mathcal{P}_2$ and containing the point $P(1, 5, 2)$.

Answers

1. (a) $\{x_1 = -1 - s, x_2 = -5 - 2s - 3t, x_3 = s, x_4 = t\}$ (b) $\{x_1 = 0, x_2 = 0, x_3 = -1, x_4 = -1\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -4 \\ 2 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -3 \\ 3 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

2. (a) $\frac{1}{48}$ (b) 3 (c) not enough information

3. (a) 4 (b) 1 (unique solution)

4. $A^T = -A \Rightarrow \det(A^T) = \det(-A) \Rightarrow \det(A) = (-1)^n \det(A)$ or $\det(A) = -\det(A)$ if n is odd $\Rightarrow \det(A) = 0$

5. (a) $\left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 1 & 1 & 1 & -6 \\ 0 & 1 & 2 & 2 \end{array}\right]$ (b) $\frac{-5}{1} = -5$

6. $k = \frac{-1}{6}$

7. (a) $A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ -D^{-1}CB^{-1} & D^{-1} \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ -15 & -24 & 1 & 2 & 0 \\ -5 & -8 & 0 & 1 & 0 \\ -5 & -8 & 0 & 0 & 1 \end{bmatrix}$

8. (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & k+18 \end{bmatrix}$ (b) $5k + 90$ (c) $k = -18$

(d) $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$

9. (a) 3 (b) $\begin{bmatrix} 19 & 20 \\ 0 & -3 \end{bmatrix} = 7 \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} - 5 \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}$ (c) No. This set has only a 2-dimensional span.

10. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ (many solutions possible)

11. (a) $A = \begin{bmatrix} \frac{7}{2} & \frac{-3}{2} \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 32 \\ 76 \end{bmatrix}$

12. (a) $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ 33 \end{bmatrix}$ (b) $T\left(3\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ 33 \end{bmatrix}$ is not equivalent to $3\,T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -9 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

13.(a) $\begin{bmatrix} 2\sqrt{3}/3 \\ -2\sqrt{3}/3 \\ -2\sqrt{3}/3 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$ (c) $\sqrt{6}$ units (d) $(3, 4, 2)$

(e) $2\sqrt{6}$ units^2

14. (a) Yes. $\det(A) = 0 \Rightarrow \det(kA) = k^n \det(A) = 0$ for any scalar k (b) No. Many counter-examples possible: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15. CANNOT, MUST, MIGHT, MUST

16. (a) $(4, -1, 6)$ (b) $\cos\theta = \frac{7\sqrt{11}}{33}$ (c) $-5x + 4y + 3z = -6$ (d) $(\frac{6}{5}, 0, 0)$

17. $\frac{1}{2}$ units

18. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$