1. Given events A, B and C such that $(A \cup B)$ and C are *disjoint*, and given the probabilities

$$P(A) = 0.4 \qquad P(A \cup C) = 0.5 \qquad P(A' \cap B) = 0.15$$

find each probability.

  (a) (1 point) $P(C)$

  (b) (2 points) $P(A \cup B)$

  (c) (2 points) $P(A \cup B \cup C')$

  (d) (2 points) $P(A \mid A \cup C)$

2. If you roll three fair dice, what is the probability that

  (a) (2 points) all three dice show the same number?

  (b) (2 points) exactly two of the dice show the same number?

  (c) (2 points) all three dice show different numbers?

3. Eight family members line up in front of the fireplace for a picture.

  (a) (1 point) How many different arrangements of family members are possible?

  (b) (2 points) If all arrangements are equally likely, what is the probability that the grandparents Alice and Bob will be in the middle two places (positions 4 and 5)?

  (c) (2 points) Suppose the family includes four males and four females. What is the probability that the picture arrangement will alternate male, female, male, female, etc., or female, male, female, male, etc.?

4. A committee on the environment has six members from the town council. The full town council has 30 members: 13 Liberals, 9 Tories and 8 Greens.

  (a) (1 point) How many different committees on the environment are possible?

  (b) (2 points) How many different committees are possible if *at least* one of the six members is a Tory?

  (c) (2 points) What is the probability that all six committee members are from the same party?

5. Over 16 thousand degrees in mathematics were given by U.S. colleges and universities in a recent year; 73% of them were bachelor's degrees, 21% were master's degrees, and the rest were doctorates. Moreover, women earned 48% of the bachelor's degrees, 42% of the master's degrees, and 29% of the doctorates.

  (a) (4 points) You choose one of these 16 thousand degrees at random and find that it was awarded to a woman. What is the probability that it is a bachelor's degree?

  (b) (1 point) Are the events *is a woman graduate in mathematics* and *earned a bachelor's degree in mathematics* independent? Justify your answer.

6. A dépanneur orders copies of a certain magazine for its magazine rack each week. Let X be the demand for the magazine and suppose the probability distribution of X is

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$

  (a) (3 points) Calculate the expected value $E(X)$ and variance $V(X)$.

  (b) (3 points) Suppose the store owner actually pays \$2.00 for each copy of the magazine and the price to customers is \$4.00. If the owner orders six copies of the magazine and copies left unsold at the end of the week are wasted, express the net revenue Y as a function of the demand X and find the expected net revenue $E(Y)$ and variance $V(Y)$.

7. A factory produces individual cups of fruit salad. Each cup is filled with 15 pieces of fruit, taken randomly from a very large tank containing 20% of cherry pieces. A fruit cup is judged acceptable if it contains at least 2 pieces of cherry.

  (a) (2 points) Find the probability that a random fruit cup is acceptable.

  (b) (2 points) A bulk package contains 48 fruit cups. Find the probability that such a package contains exactly 40 acceptable fruit cups.

8. (3 points) A group of children is composed of 20 boys (2 of them are not vaccinated) and 35 girls (5 of them are not vaccinated). A sample of 10 children is obtained by randomly selecting 5 boys and 5 girls. What is the probability that the sample will contain exactly one child that is not vaccinated?

9. The number of asteroids (with a diameter of at least $1\,\mathrm{km}$) colliding with Earth follows a Poisson distribution with an average rate of 2 asteroids per one million years.

   (a) (2 points) Find the probability that exactly 1 asteroid will hit Earth in the next 100,000 years.

   (b) (2 points) Consider the next one million years starting today. Find the probability that no asteroid will collide with Earth during the first half of this time period and that exactly two asteroids will hit the planet during the second half of this time period.

10. (4 points) In the United States, 0.6% of the population is allergic to peanuts. A random sample of 2500 people will be selected. We wish to calculate the probability that at least 10, but at most 20 people are allergic to peanuts in this sample. Use a normal approximation to find this probability, and verify that it is valid to do so.

11. Consider the continuous random variable X with the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 1 \leqslant x \leqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (2 points) Calculate the expected value of this random variable.

   (b) (3 points) Find the median (50th percentile) of this distribution.

12. An oceanographer uses a sonar system to estimate the depth of water. Suppose the sonar system is tested in an area with a constant depth of exactly $1000\,\mathrm{m}$. The measurements provided by the sonar are random but follow a normal distribution with a mean of $921\,\mathrm{m}$ and a standard deviation of $40\,\mathrm{m}$.

   (a) (2 points) Calculate the probability that the sonar will indicate a measurement above $1000\,\mathrm{m}$ in the testing area.

   (b) (2 points) The largest 7% of measurements indicate a depth in the testing area of how many metres at least?

13. (3 points) Suppose the average weight of airline passengers is $65\,\mathrm{kg}$ with a standard deviation of $10\,\mathrm{kg}$. A random sample of 81 people board a plane. If the plane only allows a maximum total weight of $5500\,\mathrm{kg}$ for its passengers, what is the probability that this limit will be exceeded? Carefully justify any result or formula that you are using.

14. (3 points) In 2009, a survey was conducted with 1018 adults in the United States with the question:

   It is now 40 years since the United States first landed men on the moon. Do you think the space program has brought enough benefits to this country to justify its costs? (Source: gallup.com)

   A total of 590 people answered "yes" to this question. Construct a 99.5% confidence interval for the true proportion of all adults in the United States who would answer "yes" to this survey question.

15. A rare congenital disease, Everley's syndrome, generally causes a reduction in concentration of blood sodium. A doctor has measured the blood sodium concentration for 16 patients with the disease. Assuming that the concentrations are normally distributed, a 95% confidence interval for the average blood sodium concentration was found to be $[108.607, 121.393]$ (in mmol/l).

   (a) (2 points) Find the sample mean $\overline{x}$ and the sample standard deviation s.

   (b) (2 points) Construct a 90% confidence interval for the standard deviation of the blood sodium concentration.

   (c) (1 point) Interpret your answer from part (b).

16. (4 points) Consider testing a single population mean with alternative hypothesis $H_a \colon \mu < 30$. Assume you have prior information that the population standard deviation is $\sigma = 8$. A sample of 100 is taken from this population. Assume that the population mean is actually $\mu = 28$. Using $\alpha = 0.05$, find the probability of a type II error.

17. (5 points) Do employees perform better at work with music playing? The music was turned on during the working hours of a business with 22 employees. Their productivity level averaged 5.2 with a standard deviation of 2.4. On a different day the music was turned off and there were 30 workers. These workers' productivity level averaged 4.8 with a standard deviation of 1.2. Assume that productivity levels with and without music playing are normally distributed.

   Test at the 0.05 level of significance whether or not the employees' mean productivity level is greater with music playing. Use the rejection region approach.

18. Does your income level affect your marital happiness? A sample was taken and each family was categorized according to income level and degree of marital happiness. The following table was obtained:

	Marital happiness		
Income	Not happy	Moderately happy	Very happy
Above average	6	115	256
Average	18	256	442
Below average	13	136	155

(a) (1 point) State the null and alternative hypotheses of the test.

(b) (3 points) Given that the chi-square test statistic is found to be 22.2, test the hypothesis at the 0.05 level of significance using the P-value approach.

(c) (1 point) Interpret your conclusion in context.

19. Suppose that a random sample of 600 entering students in a nursing program in 1989 showed 74% were still enrolled 2 years later. Another random sample of 600 entering students in 1999 showed that 66% were still enrolled 2 years later.

(a) (3 points) Find a 99% confidence interval for the difference between the true proportions.

(b) (2 points) Suppose we conduct a test with the alternative hypothesis that there is a significant difference between the true proportions. The test statistic would give the value $z^* = 3.02$. Find the corresponding P-value.

20. (5 points) A company is comparing methods for producing pipes and wants to choose the method with the least variability. A technician claims that method 2 produces pipes with less variability. He has taken a sample of the lengths of the pipes using both methods as shown below.

Method 1: 4.7 3.7 3.2 3.1 3.9 4.8 3.1 5.1 4.9 5.3 2.8 4.2 4.1
Method 2: 3.7 4.1 3.5 5.5 4.1 4.7 4.9 3.5 3.9 3.7 4.8 5.0 5.3 4.4 4.6

$$n_1 = 13 \qquad\qquad \overline{x}_1 = 4.069231 \qquad\qquad s_1^2 = 0.718974$$
$$n_2 = 15 \qquad\qquad \overline{x}_2 = 4.38 \qquad\qquad s_2^2 = 0.424571$$

Assuming lengths of pipes are normally distributed, use a 0.01 level of significance to test if the technician's claim is true. Follow the rejection region approach.

21. Suppose you have the following data for a particular bird species and you want to use this data to predict the age of a newly caught bird that has a wing length of 4.0 centimeters.

Wing length (cm)	1.5	2.2	3.1	3.2	3.2	3.9	4.1	4.7	5.2
Age (days)	4	5	8	9	10	11	12	14	16

$$\sum x_i = 31.1, \quad \sum x_i^2 = 118.33, \quad \sum y_i = 89, \quad \sum y_i^2 = 1003, \quad \sum x_i y_i = 343.7$$

(a) (3 points) Find the least squares regression line.

(b) (1 point) Use the line to estimate the age of this newly caught bird that has a wing length of 4.0 centimeters.

(c) (3 points) Calculate the correlation coefficient and comment on the accuracy of the estimate obtained in part (b).

ANSWERS

1. (a) 0.1   (b) 0.55   (c) 0.9   (d) 0.8

2. (a) 0.0278   (b) 0.4167   (c) 0.5556

3. (a) $8! = 40,320$ (b) 0.0357   (c) 0.0286

4. (a) $C_6^{30} = 593,775$ (b) $539,511$ (c) 0.0031

5. (a) 0.7684   (b) $P(\text{bachelor}|\text{women}) \neq P(\text{bachelor})$ therefore not independent

6. (a) $E(X) = 3.8, \ V(X) = 2.0267$
   (b) $Y = 4X - 12, \ E(Y) = 3.20, \ V(Y) = 32.4272$

7. (a) 0.833   (b) 0.1529

8. 0.4066

9. (a) 0.1637   (b) 0.0677

10. 0.8444

11. (a) 2.3333   (b) 2.25

12. (a) 0.0239   (b) 980.2

13. 0.0045

14. $(0.5361, 0.6231)$

15. (a) $\bar{x} = 115, \ s = 12$ (b) $(9.2959, 17.2476)$
    (c) We are 90% confident that the true standard deviation of blood sodium concentration is between 9.2959 mmol/l and 17.2476 mmol/l.

16. 0.1949

17. $H_0 : \mu_m - \mu_{nm} = 0 \qquad H_a : \mu_m - \mu_{nm} > 0$
    rejection region: $t > 1.701 \ (\nu = 28)$
    test statistic: $t^* = 0.7187$
    fail to reject H_0

18. (a) $H_0 :$ income level and marital happiness are independent
       $H_a :$ income level and marital happiness are dependent

    (b) P-value $< 0.005 \quad \Rightarrow \quad$ reject H_0

    (c) There is sufficient evidence to claim that income level and marital happiness are *dependent*, at the 5% significance level.

19. (a) $(0.0121, 0.1479)$ (b) 0.0026

20. $H_0 : \sigma_1^2 = \sigma_2^2 \qquad H_a : \sigma_1^2 > \sigma_2^2$
    rejection region: $f > 3.80$
    test statistic: $f^* = 1.6934$
    fail to reject H_0

21. (a) $Y = -1.6133 + 3.3286X$

    (b) 11.7011 days

    (c) $r = 0.9896$. This is a very strong correlation (close to 1), and therefore the prediction made in part (b) should be very accurate.