(Marks)

(4) 1. Find the third degree Taylor polynomial $T_3(x)$ for $f(x) = \sqrt{x}$ centered at $x = 4$. Use $T_3(5)$ to estimate $\sqrt{5}$, and Taylor's Inequality/Formula to estimate the accuracy of this estimate.

(5) 2. Let $f(x) = \displaystyle\int_0^x t\sqrt{t}\sin(\sqrt{t})\,dt$

(a) Find the Maclaurin series for $f(x)$; express your answer in Σ notation.

(b) Use this series to approximate $f(0.1)$ to within 10^{-7}. Justify the correctness of your approximation.

(5) 3. Use the Binomial theorem to obtain the Maclaurin series for $\dfrac{1}{\sqrt{1-x^2}}$.

(a) What is the interval of convergence of this series?

(b) Using this series, find the Maclaurin series of $\arcsin(x)$. *(Remember* $\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$.*)*

(c) What is the radius of convergence for this series?

(d) Finally, use this series to obtain an infinite series whose sum is π. *(Hint: try to find such a series whose sum is* $\frac{\pi}{6}$ *first, then multiply it by 6.)*

(8) 4. Consider the following polar curves: $r_1 = \cos\theta$ and $r_2 = \sin 2\theta$.

(a) Sketch the graphs on the same axes.

(b) Find all points of intersection (in Cartesian coordinates).

(c) Set up, but do **not** evaluate, an integral expression to find the area common to both curves.

(d) Set up and evaluate an integral expression to find the length of the first curve, r_1. Explain why you knew this value before evaluating the integral.

(8) 5. Given the curve $\mathcal{C}$ with parametric equations $x = t^3 + 1$, $y = t^2 - 3$:

(a) Find the x and y-intercepts.

(b) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Simplify your answers.

(c) Locate all points where the tangent is horizontal or vertical (identify which is which).

(d) Sketch the curve showing all these points and the intercepts, and indicate with an arrow the direction of increasing t values (the orientation).

(e) Find the area of the region below the x-axis and above the curve.

(f) Find the arc length of the section of the curve that lies below the x-axis (*i.e.* the length of the curve between its x-intercepts).

(9) 6. Sketch and name each of the following surfaces in $\mathbb{R}^3$. Show all relevant work.

(a) $2r^2 - z = 4$ (b) $\rho = 2\cos\varphi$ (c) $y = (z - x)(z + x)$

(8) 7. A particle P moves along a curve $\boldsymbol{r}(t) = \sin(t)\cos(t)\,\boldsymbol{i} + \sin^2(t)\,\boldsymbol{j} + t\,\boldsymbol{k}$.

(a) Calculate the length of the curve from $t = 0$ to $t = 2\pi$.

(b) Find the unit tangent vector $\boldsymbol{T}(t)$, the unit normal vector $\boldsymbol{N}(t)$, the curvature $\kappa(t)$, and the tangential and normal components a_T, a_N of acceleration.

Hint: You might find the double angle formulas make this simpler—though it can be done without them.

(Marks)

(5) 8. Let $z = f(x, y) = \dfrac{x + y^2}{xy}$.

(a) Find the total differential dz.

(b) Find the tangent plane to the surface $z = f(x, y)$ at $(-1, 1)$.

(c) Calculate the linear approximation dz to $\Delta z = f(Q) - f(P)$, where $P = (-1, 1)$ and $Q = (-0.9, 1.05)$, and so estimate $f(-0.9, 1.05)$.

(3) 9. Let $z = f(x, y)$ and $z = g(x, y)$ be two surfaces which intersect at the origin, so that f and g are differentiable at the origin. Show that the tangent planes to the two surfaces at the origin are perpendicular if and only if $\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial y} = -1$ at the origin.

(6) 10. Calculate the following limits; if a limit does not exist, say so (and mention $\pm\infty$ if appropriate).

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{\cos(x + y)}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + 2y^4}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2 + 2y^4}$

Be sure to justify your answers.

(3) 11. Suppose $f(x, y)$ is a differentiable function, with the property that for any t, $f(tx, ty) = t^2 f(x, y)$. Calculate $\frac{\partial}{\partial t}(f(tx, ty))$. There are two ways you could do this: do both!
From this, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2 f(x, y)$.

(5) 12. Given the (level) surface (sphere) $\mathcal{S}$: $f(x, y, z) = x^2 + y^2 + z^2 = 14$ and the point $P(1, 2, 3)$, find:

(a) the directional derivative of f at the point P in the direction of $\boldsymbol{v} = \langle 2, 1, 3\rangle$;

(b) the maximum rate of change in f at P; and

(c) the parametric equations of the tangent line at P to the curve of intersection of $\mathcal{S}$ and the plane given by $x + y + z = 6$.

(5) 13. Use Lagrange multipliers to find the surface area of a rectangular box with no top whose total volume is 10cm^3 and whose total surface area (of its 5 faces) is as small as possible.

(6) 14. Find and classify the critical points of $f(x, y) = 3xy - x^2y - xy^2$.

(6) 15. (a) Evaluate $\displaystyle\int_0^1 \int_{x^{1/3}}^1 \sqrt{1 - y^4}\, dy\, dx$ (b) Rewrite the integral $\displaystyle\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\, dy\, dx$ in the order $dx\, dy\, dz$.

(4) 16. Let $\mathcal{R}$ be the region above the $xy-$plane, and under the paraboloid $z = 1 - x^2 - 2y^2$. Set up an appropriate integral to calculate the volume of $\mathcal{R}$. (You do not have to evaluate the integral.)

(6) 17. Let $\mathcal{H}$ be the top half of the sphere $x^2 + y^2 + z^2 = 1$ (*i.e.* above $z = 0$ and inside the sphere). Calculate $\iiint_{\mathcal{H}}(2 - \sqrt{x^2 + y^2 + z^2}\,)\ dV$.

(4) 18. Let $\mathcal{D}$ be the wedge-shaped region bounded as follows: above $y = 0$, below $y = x$, and inside $x^2 + 4y^2 = 4$. Evaluate $\displaystyle\iint_{\mathcal{D}} \frac{y}{x}\, dxdy$. *Hint: Use the change of variable $u = x^2 + 4y^2$ and $v = y/x$.*

Answers

1. $T_3 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$; $T_3(5) = 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} = 2.236328$
$|R_3(5)| \le \frac{15}{16} \cdot 3^{-7/2} \cdot \frac{1}{24} = 0.000835$; so $T_3(5) = 2.236328 \pm 8.35 \times 10^{-4}$;

2. (a) $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \mp \cdots$; $t^{3/2}\sin\sqrt{t} = t^2 - \frac{1}{3!}t^3 + \frac{1}{5!}t^4 \mp \cdots$; So

$$\int_0^x t^{3/2}\sin\sqrt{t}\,dt = \frac{1}{3}x^3 - \frac{1}{4\cdot 3!}x^4 + \frac{1}{5\cdot 5!}x^5 \mp \cdots = \sum_{n=3}^{\infty}(-1)^{n+1}\frac{x^n}{(2n-5)!n}$$

(b) $f(0.1) = \frac{1}{3}0.1^3 - \frac{1}{4\cdot 3!}0.1^4 \pm \frac{1}{5\cdot 5!}0.1^5 = 0.0003291667 \pm 1.6 \times 10^{-8}$

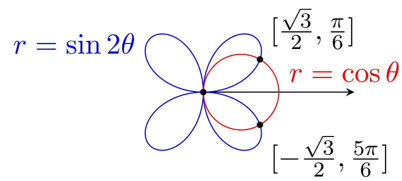
3. $$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x^2)^3 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n\,n!}x^{2n}$$

(a) Interval of convergence: $(-1, 1)$

(b) $\arcsin(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1-x^2}} = x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n\,n!\,(2n+1)}x^{2n+1}$

(c) Radius of convergence: 1 (d) $\frac{\pi}{6} = \arcsin(\frac{1}{2})$ so $\pi = 3 + \displaystyle\sum_{n=1}^{\infty}\frac{6(2n-1)!!}{2^{3n+1}n!(2n+1)}$

4. (a): Graph at right



(b) Intersections: $(0,0), (\frac{3}{4}, \pm\frac{\sqrt{3}}{4})$ (in polar at right:)

(c) $A = 2\left(\frac{1}{2}\int_0^{\pi/6}\sin^2 2\theta\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}\cos^2\theta\,d\theta\right)$

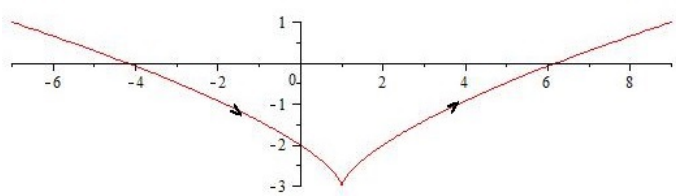
(d) $l = \int_0^{\pi}\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta = \pi$
(= circumference of circle with radius $\frac{1}{2}$: $2\pi(\frac{1}{2}) = \pi$).

5. (a) y-intercepts: $(0, -2)$ @ $t = -1$; x-intercepts: $(1 \pm 3\sqrt{3}, 0)$ @ $t = \pm\sqrt{3}$

(b) $\frac{dy}{dx} = \frac{2}{3t}$ and $\frac{d^2y}{dx^2} = -\frac{2}{9t^4}$

(c) No HT; VT at $(1, -3)$ @ $t = 0$.

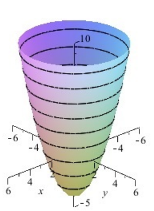
(d) Graph at right

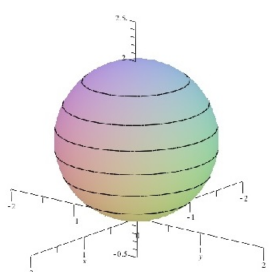


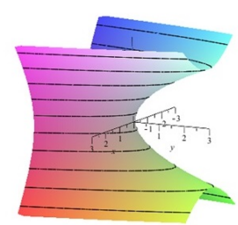
(e) $A = \int_{-\sqrt{3}}^{\sqrt{3}} -(t^2-3)(3t^2)\,dt = \frac{36}{5}\sqrt{3}$

(f) $s = \int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{9t^4 + 4t^2}\,dt$
$= \int_{-\sqrt{3}}^{\sqrt{3}} t\sqrt{9t^2+4}\,dt = \frac{2}{27}(31\sqrt{31} - 8)$

6. Three graphs: (a) a circular paraboloid (b) a sphere (c) A hyperbolic paraboloid







7. (a) $\boldsymbol{v} = \langle \cos 2t, \sin 2t, 1\rangle$ so $v = \sqrt{2}$, so $s = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$
   (b) $\boldsymbol{T}(t) = \frac{1}{\sqrt{2}}\langle \cos 2t, \sin 2t, 1\rangle$; $\boldsymbol{N}(t) = \langle -\sin 2t, \cos 2t, 0\rangle$; $\kappa = 1$; $a_T = 0$; $a_N = 2$

8. (a) $dz = -\frac{y}{x^2}\, dx + (\frac{1}{x} - \frac{1}{y^2})\, dy$
   (b) $f(-1,1) = 0$; @ $(-1,1,0) : \frac{\partial z}{\partial x} = -1, \frac{\partial z}{\partial y} = -2$, so the tangent plane is $x + 2y + z = 1$
   (c) $\Delta z \approx dz = (-1)(-0.9+1) + (-2)(1.05-1) = -0.2$ so $f(-0.9, 1.05) \approx -0.2$

9. The two normals are $\langle f_x, f_y, -1\rangle, \langle g_x, g_y, -1\rangle$ and are perpendicular if their dot product is 0, so: $f_x g_x + f_y g_y + 1 = 0$ (qed).

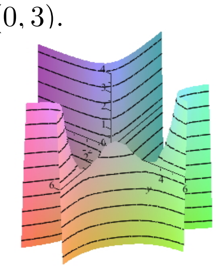
10. (a) 0 (plug in) (b) DNE (consider paths $x = 0, y = x$ *e.g.*) (c) 0 (squeeze theorem)

11. $\frac{\partial}{\partial t}(f(tx,ty)) = xf_x(tx,ty) + yf_y(tx,ty) = \frac{\partial}{\partial t}(t^2 f(x,y)) = 2tf(x,y)$. Let $t = 1$: $xf_x + yf_y = 2f$

12. (a) $\nabla(f) = \langle 2x, 2y, 2z\rangle = \langle 2,4,6\rangle$ @ P. $\boldsymbol{u} = \frac{\boldsymbol{v}}{v} = \frac{1}{\sqrt{14}}\langle 2,1,3\rangle$, so $f_{\boldsymbol{u}} = \frac{26}{\sqrt{14}}$
    (b) max rate $= |\nabla(f)(P)| = \sqrt{56}$
    (c) $\boldsymbol{n} = \langle 1,2,3\rangle \times \langle 1,1,1\rangle$ is parallel to $\langle 1,-2,1\rangle$ so the equations are $\{x = 1+t, y = 2-2t, z = 3+t\}$.

13. $V = xyz = 10$; $A = xy + 2xz + 2yz$; $\{\nabla A = \lambda \nabla V; V = 10\}$. Solving these equations gives $x = y = \sqrt[3]{20}, z = \frac{1}{2}\sqrt[3]{20}$.

14. $f_x = 3y - 2xy - y^2 = 0$; $f_y = 3x - x^2 - 2xy = 0$ so four solutions: $(0,0), (1,1), (3,0), (0,3)$.
    $D = 4xy - (3 - 2x - 2y)^2$: @ $(1,1)$ a max; @ $(0,0), (3,0), (0,3)$: saddles



15. (a) $= \displaystyle\int_0^1 \int_0^{y^3} \sqrt{1-y^4}\, dx\, dy = -\frac{1}{6}(1-y^4)^{3/2}\Big]_0^1 = \frac{1}{6}$
    (b) $= \displaystyle\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$

16. $$\int_{-1}^{1} \int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}} \int_0^{1-x^2-2y^2} dz\, dy\, dx$$

17. $$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (2-\rho)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi \int_0^{\pi/2} \sin\varphi\, d\varphi \int_0^1 (2\rho^2 - \rho^3)\, d\rho = \frac{5\pi}{6}$$

18. $= \int_0^1 \int_0^4 \frac{v}{2+8v^2}\, du\, dv = \frac{1}{4}\ln 5$