1. [8 points] Solve the following systems or show that the system has no solution. Give parametric solutions where applicable.

(a) $\begin{cases} 4x_1 + 28x_2 + 12x_3 + 4x_4 = -32 \\ 2x_1 + 14x_2 + 5x_3 = -13 \end{cases}$ (b) $\begin{cases} 2x + y + 14z = 9 \\ 5x + 8y + 50z = 28 \\ x + y + 8z = 5 \end{cases}$

2. [4 points] A craft knitter makes wool scarves, tuques, and mittens and sells them at the farmers market. A scarf is made from 2 ounces of wool and is sold for 12 dollars. A tuque is made from 2 ounces of wool and is sold for 15 dollars. A pair of mittens is made from 3 ounces of wool and is sold for 24 dollars. Last month, the craft knitter used a total of 34 ounces of wool and earned a total revenue of 225 dollars from the sale of that month's products.

(a) Define variables x, y, z, and set up a linear system to determine how many of each item were sold. **Do not solve.**

(b) Find all realistic solutions, given that the general solution to the problem is $\left\{ \begin{array}{l} x = 10 + \frac{1}{2}t \\ y = 7 - 2t \\ z = t \end{array} \right\}$.

3. [4 points] The following matrix represents a linear system: $\left[\begin{array}{cc|c} 10 & 20 & 10 \\ 2 & k^2 & k \end{array}\right]$

Find all values of k, if any, such that the system has
(a) a unique solution. (b) infinitely many solutions. (c) no solution.

4. [8 points] Let $A = \begin{bmatrix} 2 & 4 & -5 \\ 0 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 4 & 1 \\ 1 & -3 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$.
Find the following or state that they are undefined.
(a) $B^T + DC$ (b) $3B^2$ (c) $A(A^2)^{-1}(3A)$ (d) $(D - D^T)^{-1}$ (e) A matrix X such that $(CB)X = I$.

5. [3 points] Let $A = \begin{bmatrix} 2 & h \\ 1 & k \end{bmatrix}$. Find all values of h and k for which $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

6. [2 points] Let $A = \begin{bmatrix} -1 & a & b \\ 3 & 2 & 0 \\ b^2 & 0 & 6 \end{bmatrix}$. Find all possible values of a and b for which A is a symmetric matrix.

7. [4 points] (a) Find the inverse of the matrix $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & -6 & -3 \\ 2 & -5 & -4 \end{bmatrix}$ using row operations.

(b) Use your answer in part (a) to solve the system $\begin{cases} -x_1 + 3x_2 + 2x_3 = -3 \\ 2x_1 - 6x_2 - 3x_3 = 2 \\ 2x_1 - 5x_2 - 4x_3 = 1 \end{cases}$.

8. [4 points] Find the inverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & -1 \\ 5 & 1 & 0 \end{bmatrix}$ using the adjoint.

9. [2 points] Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 11$. Find the following or state that there is not enough information:

(a) $\begin{vmatrix} (2a+3d) & (2b+3e) & (2c+3f) \\ g & h & i \\ d & e & f \end{vmatrix}$ (b) $\begin{vmatrix} (a-b) & (b-a) & c \\ (d-e) & (e-d) & f \\ (g-h) & (h-g) & i \end{vmatrix}$

10. [3 points] Suppose A, B, and C are 5×5 matrices with $\det(A) = -2$, $\det(B) = 10$ and $\det(C) = 0$. Find the following or state that there is not enough information:

(a) $\det(10B^{-1})$

(b) $\det(A^5 B^T)$

(c) $\det(CA + CB)$

11. [4 points] Use Cramer's rule to solve for x_4 in the following system: $\begin{cases} x_1 + 2x_2 + x_3 + 2x_4 = 3 \\ x_1 + 2x_2 + x_3 + 4x_4 = 0 \\ x_1 + 3x_2 + x_3 + 6x_4 = 0 \\ -x_1 + 4x_2 + x_3 + 8x_4 = 0 \end{cases}$.

12. [3 points] Given $A = \begin{bmatrix} 1 & 2 & k \\ 1 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, find all value(s) of k such that:

(a) $\text{Col}(A) = \mathbb{R}^3$.

(b) The columns of A are linearly dependent.

13. [3 points] Let $\mathbf{u} = (4, -4, k)$ and $\mathbf{v} = (-6, 6, 9)$.

(a) Find all value(s) of k so that $\mathbf{u}$ and $\mathbf{v}$ are parallel.

(b) Find all value(s) of k so that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(c) Find all value(s) of k so that $||\mathbf{u}|| = ||\mathbf{v}||$.

14. [6 points] Given the point $A(1, 2, 1)$ and the vector $\mathbf{v} = (2, 1, 5)$:

(a) Find the point B such that $\overrightarrow{AB} = \mathbf{v}$.

(b) Find the unit vector $\mathbf{u}$ in the opposite direction of $\mathbf{v}$.

(c) Find a vector equation for the line L which passes through A and is parallel to $\mathbf{v}$.

(d) True or False?: The line L is a subspace of $\mathbb{R}^3$.

(e) Find an equation of the plane P (in the form $ax + by + cz = d$) that contains $(0, 0, 0)$ and has normal vector $\mathbf{v}$.

(f) True or False?: The plane P is a subspace of $\mathbb{R}^3$.

15. [4 points] Let $S = \{(x, y, z) \in \mathbb{R}^3 | xy = z\}$.

(a) Is $\mathbf{0} \in S$?

(b) Give two nonzero vectors in S.

(c) Is S a subspace of $\mathbb{R}^3$? Justify your answer.

16. [11 points] The matrix $U = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 2 & 1 & 3 & 2 & 7 \\ 3 & 1 & a & b & 5 \end{bmatrix}$ reduced to $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, and $\mathbf{u}_5$ be the columns of U.

(a) Determine, with justification, whether each of the following sets is linearly independent or linearly dependent.
i. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ ii. $\{\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5\}$ iii. $\{\mathbf{u}_2, \mathbf{u}_3\}$ iv. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

(b) List the sets in part (a) that span $\mathbb{R}^3$.

(c) List the sets in part (a) that are a basis for $\mathbb{R}^3$

(d) Find the missing values a and b of the matrix U.

(e) Find a basis for $\text{Nul}(U)$ and state its dimension.

Question 16 continues on the next page.

(f) Given that $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $U\mathbf{x} = \mathbf{u}_1$, find the general solution to $U\mathbf{x} = \mathbf{u}_1$.

17. [2 points] Suppose that A is an $m \times 3$ matrix and that the nullity of A is 2.

(a) What is the rank of A^T?

(b) If the nullity of A^T is 3, find m.

18. [4 points] A simple economy consists of two industries: Goods and Services. To produce \$1 of Goods requires 10¢ of Goods and 20¢ of Services. To produce \$1 of Services requires 30¢ of Goods and 40¢ of Services.

(a) Determine whether or not each of the industries is profitable. Justify your answer.

(b) Find the production required to meet an external demand for \$720 of Goods and \$240 of Services.

19. [8 points] The *Banana Computer Company* manufactures desktops, laptops, and tablets at two factories: Factory A and Factory B. The company has an order to deliver at least 800 desktops, 700 laptops, and 900 tablets in the next year. The monthly production and operational cost of each factory is given by the following table:

	Desktops	Laptops	Tablets	Cost
Factory A	200	100	100	\$200,000
Factory B	100	100	300	\$300,000

The company would like to determine the number of months each factory should operate so as to fulfill the order at minimum cost.

(a) Name variables and set-up a linear program that describes the situation.

(b) Solve the program using the graphical method.

20. Criminologists of one province analyzed the rates of recidivism in those charged with vandalism. They noted that, if an individual is charged with vandalism on any given year, then they are 8% likely to be charged with vandalism again the following year. In contrast, of those who are *NOT* charged with vandalism one year, there is a 98% probability that they will also not be charged with vandalilsm the following year.

(a) [1 point] Provide a probability matrix P that may serve as a transition matrix for this problem.

(b) [3 points] Provide the steady-state vector $\mathbf{q}$ of the transition matrix P from part (a). Give your answer final answer in fractions.

(c) [3 points] Fred was charged with vandalism in the year 2015. According to the information provided, what is the likelihood that he will *NOT* be charged with vandalism in the year 2017?

21. [6 points] Mr. Dukakis, a retired Linear Algebra teacher has been kidnapped. The lead detective finds a paper found on the floor on which the matrix $A = \begin{bmatrix} 7 & 4 \\ 1 & 1 \end{bmatrix}$ and the letters **CWWL** have been frantically scrawled. It turns out that Mr. Dukakis was counting on someone being able to recognize this as a Hill 2-cipher and decode the name of his kidnapper. Who kidnapped Mr. Dukakis?

You may find the following table of multiplicative inverses mod (26) helpful:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

END OF EXAM. ANSWERS ON NEXT PAGE.

Question 21 continues on the next page.

ANSWERS:

1.(a) $\begin{cases} x_1 = 1 - 7s + 5t \\ x_2 = s \\ x_3 = -3 - 2t \\ x_4 = t \end{cases}$

1.(b) $\begin{cases} x = 4 \\ y = 1 \\ z = 0 \end{cases}$

2.(a) Let $x =$#scarves sold.
Let $y =$#tuques sold.
Let $z =$#pairs of mittens sold.

$$\begin{cases} 2x + 2y + 3z = 34 \\ 12x + 15y + 24z = 225 \end{cases}$$

2.(b) Two realistic solutions:
$(10, 7, 0)$ and $(11, 3, 2)$.

3.(a) $k \neq \pm 2$. 3.(b) $k = 2$ 3.(c) $k = -2$

4.(a) $\begin{bmatrix} -5 & 18 & -5 \\ 5 & -24 & 8 \end{bmatrix}$

4.(b) Undefined.

4.(c) $3I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4.(d) Undefined.

4.(e) $X = (CB)^{-1} = \begin{bmatrix} 1/6 & -1/14 \\ 1/6 & 1/14 \end{bmatrix}$

5. $h = -4, k = -2$.

6. $a = 3$, $b = 0$ or $b = 1$.

7.(a) $A^{-1} = \begin{bmatrix} 9 & 2 & 3 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

7.(b) $\begin{cases} x_1 = -20 \\ x_2 = -5 \\ x_3 = -4 \end{cases}$

8. $A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 3 \\ -18 & -3 & 12 \end{bmatrix}$

9.(a) -22 9.(b) 0

10.(a) 10000 10.(b) -320 10.(c) 0

11. $x_4 = \dfrac{\det(A_4)}{\det(A)} = \dfrac{-6}{4} = -\dfrac{3}{2}$.

12.(a) $k \neq 3/2$ 12.(b) $k = 3/2$

13.(a) $k = -6$ 13.(b) $k = 16/3$ 13.(c) $k = \pm 11$.

14.(a) $B(3, 3, 6)$

14.(b) $\mathbf{u} = \left(\frac{-2}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}\right)$.

14.(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

14.(d) False. Doesn't contain $\mathbf{0}$.

14.(e) $2x + y + 5z = 0$.

14.(f) True. A plane through $\mathbf{0}$ is a subspace.

15.(a) Yes. $0 \cdot 0 = 0$.

15.(b) $(1, 1, 1)$ and $(1, 0, 0)$ for example.

15.(c) No. It is not closed under Scalar Multiplication.
C.-E.: $\mathbf{u} = (1, 1, 1) \in S$. $2\mathbf{u} \notin S$.
Nor is it closed under vector addition.
C.-E.: $\mathbf{u} = \mathbf{v} = (1, 1, 1) \in S$, $\mathbf{u} + \mathbf{v} \notin S$.

16.(a)i. LD. $\mathbf{u}_3 = 2\mathbf{u}_1 - \mathbf{u}_2$.

16.(a)ii. LI. Matrix $[\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5]$ has rank 3.

16.(a)iii. LI. Neither is a multiple of the other.

16.(a)iv. LD. Four vectors in $\mathbb{R}^3$ are always LD.

16.(b) $\{\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

16.(c) $\{\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5\}$ only.

16.(d) $a = 5$, $b = 1$.

16.(e) $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$. $\text{Dim}(\text{Nul}(U)) = 2$.

16.(f) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

17.(a) $\text{Rank}(A^T) = 1$ 17.(b) $m = 4$.

18.(a) Both industries are profitable:
It costs 30¢ to produce \$1 in goods.
It costs 70¢ to produce \$1 in sevices.

18.(b) Produce \$1050 in goods and \$750 in services.

19.(a) Let $x =$#months Factory A operates.
Let $y =$#months Factory B operates.

Minimize $C = 200000x + 300000y$
subject to
$$\begin{aligned} 200x + 100y &\geqslant 800 \\ 100x + 100y &\geqslant 700 \\ 100x + 300y &\geqslant 900 \\ 0 \leqslant x \leqslant 12, &\quad 0 \leqslant y \leqslant 12 \end{aligned}$$

19.(b) Minimum cost of \$1.5 million occurs when Factory A operates 6 months and Factory B operates 1 month. Other corner points: $(1, 6), (0, 8), (0, 12), (12, 12), (12, 0), (9, 0)$.

20. Min $z = -24$ at $(0, 0, 8, 2, 0)$.

21.(a) In second table, no positive pivots below -5 in the x_1 column.

21.(b) $z = 5t + 2$ at $(t, t + \frac{1}{3}, 0, 2t + 5, \frac{1}{3}, 0)$;
so $z = 5002$ at $(1000, \frac{3001}{3}, 0, 2005, \frac{1}{3}, 0)$.

22.(a) $P = \begin{bmatrix} 0.08 & 0.02 \\ 0.92 & 0.98 \end{bmatrix}$

22.(b) $\mathbf{q} = (\frac{7}{23}, \frac{1}{47}, \frac{46}{47})$

22.(c) 97.52%

23. ERIC (using $A^{-1} = \begin{bmatrix} 9 & 16 \\ 17 & 11 \end{bmatrix}$)