1. (6 points) Let $A = \begin{bmatrix} 1 & 1 & c \\ 1 & c & c \\ c & c & c \end{bmatrix}$. and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ c \end{bmatrix}$. Find the value(s) of c for which

(a) $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(b) $A\mathbf{x} = \mathbf{b}$ has no solution.

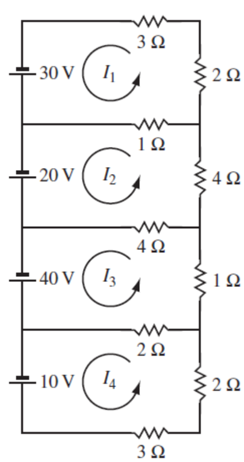
(c) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(d) $\text{Col}(A)$ is a plane.

(e) $\text{Nul}(A)$ is a plane.

2. (3 points) Given $A = \begin{bmatrix} 1 & 5 & 1 & -7 \\ -2 & -10 & 1 & 2 \\ -5 & -25 & 1 & 11 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -3 \\ -9 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$. Give your answer in parametric vector form.

3. (2 points) Set up an augmented matrix for finding the loop currents of the following electrical circuit. **You do not have to solve it.**



4. (3 points) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. Find an LU factorization of A.

5. (8 points) Let L and A be 3×3 matrices, with L being lower triangular with 1's along the main diagonal and $\det A = 5$, and let I be the 3×3 identity matrix. Compute each of the following determinants, or state that there is not enough information to do so.

(a) $\det(2A^T L)$

(b) $\det((A^{-1})^2)$

(c) $\det(L + A)$

(d) $\det(L + 2I)$

6. (4 points) Solve the following linear system for x_4 only, using Cramer's Rule.

$$\begin{aligned} - 2x_2 + 2x_3 - x_4 &= 0 \\ -3x_1 + 6x_2 + 2x_3 + x_4 &= 0 \\ -x_1 \qquad - x_3 + x_4 &= 0 \\ -2x_1 + x_2 \qquad + x_4 &= 1 \end{aligned}$$

7. (6 points) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ -3 & -3 & 3 \end{bmatrix}$.

(a) Use row reduction to find the inverse of A.

(b) Write A as a product of elementary matrices.

8. (3 points) Solve for the matrix X in $(A - AX)^{-1} = X^{-1}B$.

9. (4 points)

(a) Let M be a matrix such that $M^2 = I$. Prove that $\det M = \pm 1$.

(b) If N is a matrix such that $\det N = 1$, does N^2 necessarily equal I? Support your answer with a proof or a counterexample.

10. (5 points) Suppose that A and B are $n \times n$ matrices. Complete the sentences with the word **MUST**, **MIGHT** or **CANNOT** as appropriate.

(a) If E_1 and E_2 are two elementary matrices, then E_1E_2 __________ be equal to E_2E_1.

(b) If $A^3 = I$ then A __________ be invertible.

(c) If $\det A$ is zero then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ __________ be invertible.

(d) The expression $(I - A)(I + A)$ __________ be equal to $I - A^2$.

(e) If B has no column of zeros, but AB does, then the columns of A __________ be linearly independent.

11. (10 points) Consider the set $S = \{X \in \mathbf{M_{2\times 2}} : \; AX - X = 0\}$ where $A = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}$.

(a) Find a nonzero matrix that is in S.

(b) Find a nonzero 2×2 matrix that is not in S.

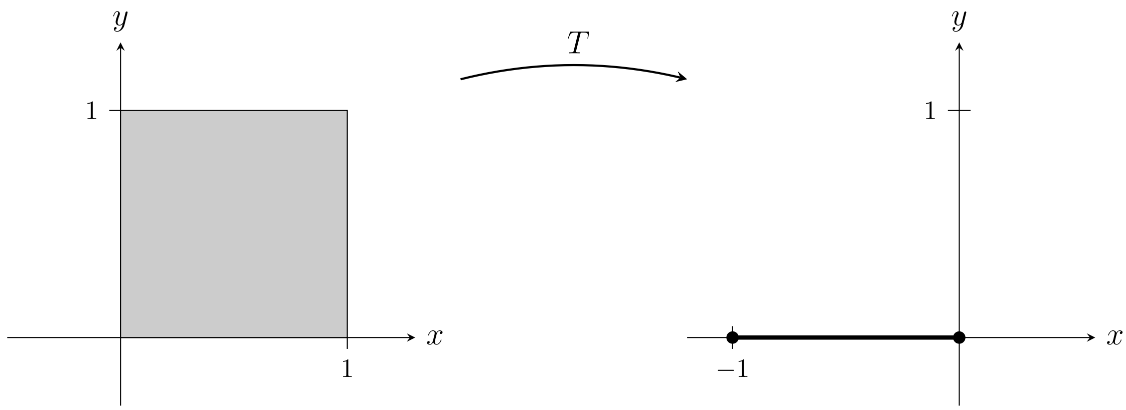
(c) Does S contain the zero element? Justify your answer.

(d) Is S closed under addition? Justify your answer.

(e) Is S closed under scalar multiplication? Justify your answer.

(f) Is S a subspace?

12. (7 points)



Let P_x be the standard matrix of the transformation that projects points onto the x axis and let P_y be the standard matrix of the transformation that projects points onto the y axis.

(a) Give the matrices P_x and P_y.

(b) Find the standard matrix R_1 of a rotation such that R_1P_x transforms the unit square on the left side into the line segment on the right.

(c) Do R_1 and P_x commute?

(d) Find the standard matrix R_2 of a rotation such that R_2P_y transforms the square into the segment.

(e) Do R_2 and P_y commute?

(f) Find a basis for the null space of R_2P_y.

13. (8 points) A matrix A and its reduced row echelon form are given below.

$$A = \begin{bmatrix} 1 & 4 & -2 & 4 & v & 3 & 6 \\ 3 & 12 & -6 & 12 & w & 2 & 15 \\ -2 & -8 & 4 & -8 & x & -1 & -13 \\ 1 & 4 & -2 & 5 & y & 0 & 3 \\ 3 & 12 & -6 & 12 & z & 3 & 10 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 4 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{a_i}$ denote the i^{th} column of A, and $\mathbf{u_j}$ denote the j^{th} column of U, so that

 Question 13 continues on the next page.

$$A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} & \mathbf{a_4} & \mathbf{a_5} & \mathbf{a_6} & \mathbf{a_7} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} & \mathbf{u_4} & \mathbf{u_5} & \mathbf{u_6} & \mathbf{u_7} \end{bmatrix}.$$

You may use the above notation in your answers to the following questions.

(a) Is $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}\}$ a basis for $\text{Col}(A)$?

(b) Find a basis for the column space of A.

(c) Find a basis for $\text{Row}(A)$.

(d) Find a basis for $\text{Nul}(A)$.

(e) Express $\mathbf{a_7}$ as a linear combination of basis vectors for $\text{Col}(A)$.

(f) Find the values of the entries v, w, x, y, and z in the matrix A.

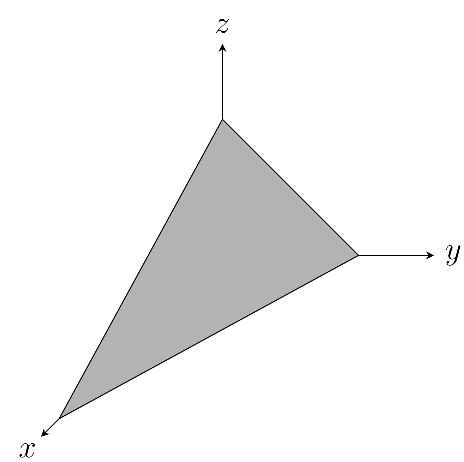
(g) What is the dimension of $\text{Nul}(A^T)$?

14. (4 points) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{q}$ be non-zero vectors in $\mathbb{R}^3$ and suppose $\text{Span}\,\{\mathbf{u}, \mathbf{v}\} = \text{Span}\,\{\mathbf{p}, \mathbf{q}\}$.

(a) If $\text{Span}\,\{\mathbf{u}, \mathbf{v}\}$ is a line explain why $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

(b) If $\text{Span}\,\{\mathbf{u}, \mathbf{v}\}$ is a plane explain why $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

15. (11 points) A triangle is created by joining the x-, y-, and z-intercepts of the plane $x + 2y + 2z = 18$. The figure is shown below.



(a) Find an equation of the line through the origin perpendicular to the plane.

(b) Find the point of intersection of the line from part (a) with the plane.

(c) Using the above results find the distance between the origin to the plane.

(d) Find the coordinates of the vertices of the triangle shown in the figure.

(e) Find the area of the triangle shown in the figure.

(f) Find the distance between the point $P(2, 2, 2)$ to the line found in part (a).

(g) If θ is the angle between the plane $x + 2y + 2z = 18$ and the xy-plane, find the value of $\cos\theta$.

16. (7 points) Let I be the $n \times n$ identity matrix and let $B = \begin{bmatrix} I & -I \\ I & I \end{bmatrix}$.

(a) Compute and simplify B^2.

(b) Find B^{-1}.

(c) Under what conditions on the $n \times n$ matrices X, Y, Z, and W will $\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$ commute with $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$.

17. (3 points) Suppose $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are vertices of a triangle in $\mathbb{R}^n$ centered at the origin so that $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$. Suppose $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are vertices of another triangle in $\mathbb{R}^n$ centered at the origin so that $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 = \mathbf{0}$. Now let T be a linear transformation such that $T(\mathbf{a}_1) = \mathbf{b}_1$ and $T(\mathbf{a}_2) = \mathbf{b}_2$. Show that $T(\mathbf{a}_3) = \mathbf{b}_3$

18. (6 points) Let $T : \mathbf{P}_3 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{p}(x)) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \end{bmatrix}$.

(a) Find a basis of for the kernel of T.

(b) Find $T(-2x^3 + 3x^2 + 5x - 6)$.

(c) Express the polynomial $-2x^3 + 3x^2 + 5x - 6$ as a linear combination of the basis polynomials from part (a).

Answers

1. (a) $c \neq 0, 1$ (b) $c = 1$ (c) $c = 0$ (d) $c = 0$ (e) $c = 1$

2. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$

3. $\left[\begin{array}{rrrr|r} 6 & -1 & 0 & 0 & 30 \\ -1 & 9 & -4 & 0 & 20 \\ 0 & -4 & 7 & -2 & 40 \\ 0 & 0 & -2 & 7 & 10 \end{array}\right]$

4. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

5. (a) 40 (b) $\frac{1}{25}$ (c) not enough information (d) 27

6. $x_4 = \frac{-22}{3}$

7. (a) $A^{-1} = \begin{bmatrix} -3 & 1 & -2/3 \\ 1 & 0 & 1/3 \\ -2 & 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

 Question 18 continues on the next page.

8. $X = (B^{-1} + A)^{-1} A$

9. (a) $\det(M^2) = \det I \Rightarrow (\det M)^2 = 1 \Rightarrow \det M = \pm 1$

(b) No. Counterexample: $N = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

10. (a) MIGHT (b) MUST (c) CANNOT
(d) MUST (e) CANNOT

11. (a) $\begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) Yes. $A\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

(d) Yes. If $X_1, X_2 \in S$, then $A(X_1 + X_2) - (X_1 + X_2) = (AX_1 - X_1) + (AX_2 + X_2) = 0 + 0 = 0$.

(e) Yes. If $X_1 \in S$, then $A(kX_1) - (kX_1) = k(AX_1 - X_1) = k(0) = 0$. (f) Yes.

12. (a) $P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $R_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) Yes.

(d) $R_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (e) No. (f) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

13. (a) No. (b) $\{\mathbf{a_1}, \mathbf{a_4}, \mathbf{a_5}, \mathbf{a_6}\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 4 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 2 \\ -3 \\ 1 \end{bmatrix} \right\}$ (e) $\mathbf{a_7} = \mathbf{a_1} - 2\mathbf{a_5} + 3\mathbf{a_6}$

(f) $v = 2, w = -3, x = 4, y = -1, z = 1$ (g) 1

14. (a) $\mathbf{u}$ and $\mathbf{v}$ parallel $\Rightarrow (\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0} \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$

(b) Span $\{\mathbf{u}, \mathbf{v}\}$ and Span $\{\mathbf{p}, \mathbf{q}\}$ describe the same plane with parallel normal vector directions $\mathbf{N_1} = \mathbf{u} \times \mathbf{v}$ and $\mathbf{N_2} = \mathbf{p} \times \mathbf{q}$, so $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{N_1} \times \mathbf{N_2} = \mathbf{0}$.

15. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ (b) $(2, 4, 4)$ (c) 6 units (d) $(18, 0, 0)$, $(0, 9, 0)$, and $(0, 0, 9)$

(e) $\frac{243}{2}$ units^2 (f) $\frac{2\sqrt{2}}{3}$ units (g) $\frac{2}{3}$ radians

16. (a) $\begin{bmatrix} 0 & -2I \\ 2I & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{-1}{2}I & \frac{1}{2}I \end{bmatrix}$ (c) $X = -Y$ and $W = Z$

17. $T(\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3}) = T(\mathbf{0}) = \mathbf{0}$ and $T(\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3}) = T(\mathbf{a_1}) + T(\mathbf{a_2}) + T(\mathbf{a_3})$ for any linear transformation T, so $T(\mathbf{a_1}) + T(\mathbf{a_2}) + T(\mathbf{a_3}) = \mathbf{b_1} + \mathbf{b_2} + T(\mathbf{a_3}) = \mathbf{0} \Rightarrow T(\mathbf{a_3}) = -\mathbf{b_1} - \mathbf{b_2} = \mathbf{b_3}$.

18. (a) $\{x^3 - 7x + 6, x^2 - 3x + 2\}$ (multiple solutions possible) (b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) $-2(x^3 - 7x + 6) + 3(x^2 - 3x + 2)$