

1. (30 points) Evaluate the following integrals.

(a)  $\int \frac{7x - 6}{(x + 1)(4x^2 + 9)} dx$

(b)  $\int e^x \sec^3(e^x) \tan^3(e^x) dx$

(c)  $\int \sqrt{5 + 4x - x^2} dx$

(d)  $\int_0^{1/6} \arccos(3x) dx$

(e)  $\int \frac{x + \cos \sqrt{x+2}}{\sqrt{x+2}} dx$

(f)  $\int \frac{1}{x^3} \sin(1/x) dx$

2. (9 points) Calculate the following limits.

(a)  $\lim_{x \rightarrow 0^+} (e^x + x)^{2/x}$

(b)  $\lim_{x \rightarrow \infty} x \sin\left(\frac{\pi}{x}\right)$

(c)  $\lim_{x \rightarrow \infty} [\ln(3x^2 + 5) - \ln(x^2 + 1)]$

3. (8 points) Determine whether the following improper integrals converge or diverge. If an integral converges, give its exact value.

(a)  $\int_0^{\infty} (1 - x)e^{-x} dx$

(b)  $\int_0^{\pi/2} \frac{\sin x}{1 - \cos x} dx$

4. (5 points) Find the area of the region enclosed by the curves  $4x + y^2 = 0$  and  $y = 2x + 4$ .

5. (6 points) Let  $\mathcal{R}$  be the region bounded by  $y = x^3$ , the  $y$ -axis, and  $y = 8$ . Set up, but **do not evaluate**, an integral for the volume of the solid obtained by rotating  $\mathcal{R}$  around:

(a) the  $x$ -axis

(b) the  $y$ -axis

(c) the line  $x = -2$

6. (5 points) Find the length of the curve  $y = \frac{1}{2}(e^x + e^{-x})$  from  $x = 0$  to  $x = 1$ .

7. (5 points) Solve the differential equation

$$\cos^2 x \frac{dy}{dx} + y \tan x = 0$$

with initial condition  $y(0) = e$ . Express  $y$  as a function of  $x$ .

8. (5 points) Consider the sequence  $\{a_n\} = \left\{ \frac{3n + \ln n}{2 - 5n} \right\}$ .

(a) Does the sequence  $\{a_n\}$  converge and, if so, to what value?

(b) Does the corresponding series  $\sum_{n=1}^{\infty} \frac{3n + \ln n}{2 - 5n}$  converge? Justify your answer.

9. (9 points) Determine whether the following series converge or diverge. State which test you are using and display a proper solution.

(a)  $\sum_{n=1}^{\infty} \frac{n!(n+1)! 3^n}{(2n)!}$

(b)  $\sum_{n=1}^{\infty} \frac{3 + \cos n}{\sqrt{n}}$

(c)  $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^5 + 3n^2 + 7}}$

10. (7 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answer.

(a)  $\sum_{n=1}^{\infty} (-1)^n \left( \frac{2n+1}{3n+4} \right)^{2n}$

(b)  $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

11. (3 points) Find the sum of the series  $\sum_{n=2}^{\infty} (-1)^n \frac{3}{2^n}$ . Justify your answer.

12. (4 points) Find the radius and interval of convergence of the power series  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(3n+1)4^n} (x+2)^n$ .

13. (4 points) Let  $f(x) = \frac{1}{(1+x)^3}$ .

(a) Write the first five non-zero terms of the Maclaurin series for  $f(x)$ .

(b) Find the formula for the  $n$ th term of the series and express the series in sigma ( $\Sigma$ ) notation.

## ANSWERS

1. (a)  $\frac{1}{2} \ln(4x^2 + 9) - \ln|x + 1| + \frac{1}{2} \arctan(\frac{2}{3}x) + C$   
 (b)  $\frac{1}{5} \sec^5(e^x) - \frac{1}{3} \sec^3(e^x) + C$   
 (c)  $\frac{9}{2} \arcsin(\frac{1}{3}(x - 2)) + \frac{1}{2}(x - 2)\sqrt{9 - (x - 2)^2} + C$   
 (d)  $x \arccos(3x) - \frac{1}{3}\sqrt{1 - 9x^2} \Big|_0^{1/6} = \frac{1}{18}\pi + \frac{1}{3} - \frac{1}{6}\sqrt{3}$   
 (e)  $\frac{2}{3}(x + 2)^{3/2} - 4\sqrt{x + 2} + 2 \sin \sqrt{x + 2} + C$   
 $= \frac{2}{3}(x - 4)\sqrt{x + 2} + 2 \sin \sqrt{x + 2} + C$

(f)  $\frac{1}{x} \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) + C$

2. (a)  $e^4$  (b)  $\pi$  (c)  $\ln 3$

3. (a) Converges to 0  
 (b) Diverges (to  $\infty$ )

4.  $\int_{-4}^2 \left[-\frac{1}{4}y^2 - \frac{1}{2}(y - 4)\right] dy = 9$

5. For each part of this question, the first answer expresses the required volume as an integral using washers (or disks), and the second answer expresses the volume using cylindrical shells.

(a)  $\int_0^2 \pi[8^2 - (x^3)^2] dx = \int_0^8 2\pi y \sqrt[3]{y} dy$

(b)  $\int_0^8 \pi y^{2/3} dy = \int_0^2 2\pi x(8 - x^3) dx$

(c)  $\int_0^8 \pi[(\sqrt[3]{y} + 2)^2 - 2^2] dy = \int_0^2 2\pi(x + 2)(8 - x^3) dx$

6.  $\frac{1}{2}(e - 1/e)$

7.  $y = e^{1 - \frac{1}{2} \tan^2 x} = e^{\frac{3}{2} - \frac{1}{2} \sec^2 x}$

8. (a)  $\{a_n\}$  converges to  $-\frac{3}{5}$ .

- (b) Since  $\lim_{n \rightarrow \infty} a_n \neq 0$ , the series  $\sum_{n=1}^{\infty} a_n$  diverges by the divergence test.

9. Let  $a_n$  be the  $n$ th term of the series in question.

- (a) Converges by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{3(n+1)(n+2)}{(2n+2)(2n+1)} = \frac{3(n+2)}{2(2n+1)} \rightarrow \frac{3}{4} < 1$$

- (b) Diverges by the direct comparison test: since  $\cos n \geq -1$  for all  $n$ ,

$$a_n \geq \frac{2}{\sqrt{n}} = b_n$$

and  $\sum b_n$  diverges because it is a non-zero multiple of a divergent  $p$ -series ( $p = \frac{1}{2} \leq 1$ ).

- (c) Since

$$a_n = \frac{n(1 + 1/n)}{\sqrt{n^5} \sqrt{1 + 3/n^3 + 7/n^5}} = \frac{1}{n^{3/2}} \frac{1 + 1/n}{\sqrt{1 + 3/n^3 + 7/n^5}}$$

use the limit comparison test with the convergent  $p$ -series  $\sum b_n = \sum 1/n^{3/2}$ :

$$\frac{a_n}{b_n} = \frac{1 + 1/n}{\sqrt{1 + 3/n^3 + 7/n^5}} \rightarrow 1 \neq 0, \infty$$

and so  $\sum a_n$  converges.

10. Let  $a_n$  be the  $n$ th term of the series in question.

- (a) Converges absolutely by the root test:

$$|a_n|^{1/n} = \left(\frac{2n+1}{3n+4}\right)^2 \rightarrow \frac{4}{9} < 1$$

- (b) Converges conditionally.  $\sum |a_n|$  diverges by direct comparison with  $\sum 1/\sqrt{n}$  since

$$\frac{\ln n}{\sqrt{n}} > \frac{1}{\sqrt{n}} \quad \text{if } n \geq 3$$

(alternately, you can use the integral test). On the other hand,  $(\ln n)/\sqrt{n} \rightarrow 0$  as  $n \rightarrow \infty$  and is decreasing because

$$\frac{d}{dx} \left[ \frac{\ln x}{\sqrt{x}} \right] = \frac{2 - \ln x}{2x^{3/2}} < 0 \quad \text{if } x > e^2,$$

and so  $\sum a_n$  converges by the alternating series test.

11. This is a geometric series with  $r = -\frac{1}{2}$  and  $a = \frac{3}{4}$ .

Since  $|r| = \frac{1}{2} < 1$ , the series converges to

$$\frac{a}{1 - r} = \frac{1}{2}$$

12.  $R = 4, (-6, 2]$

13.  $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2}(n+1)(n+2)x^n$   
 $= 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \dots$