1. (10 points) Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{x+4}-\frac{1}{3x}}{x-2}$

(b) $\displaystyle\lim_{x\to \frac{\pi}{3}^+} \frac{\sqrt{x}}{2\cos(x)-1}$

(c) $\displaystyle\lim_{x\to -\infty} \frac{9-3x}{2x-\sqrt{4x^2+3x-9}}$

(d) $\displaystyle\lim_{x\to \frac{1}{3}} \frac{9x^2-1}{8x-3|x^2-1|}$

(e) $\displaystyle\lim_{x\to 0^+} e^{-5/x}\sin\left(\frac{5}{x}\right)$

2. (5 points) Let

$$f(x) = \begin{cases} x^2+6x-18 & \text{if } x<b, \\ a & \text{if } x=b, \\ 5x-6 & \text{if } x>b. \end{cases}$$

Find all pairs of values a and b so that f is continuous everywhere.

3. (3 points) Draw the graph of a function f that has **all** of the following properties:

- the graph of f has horizontal asymptotes given by $y=-2$ and $y=2$,
- $\displaystyle\lim_{x\to 3} f(x)$ exists but f is not continuous at 3,
- f is continuous but not differentiable at $x=-1$.

4. Consider the function $f(x)=\dfrac{2}{\sqrt{4-3x}}$.

(a) (4 points) Find $f'(x)$ by using the limit definition of the derivative.

(b) (2 points) Find $f'(x)$ by using the rules of differentiation, and check that your answer matches with part (a).

(c) (2 points) Find an equation of the tangent line to the graph of f at the point where $x=1$.

5. (12 points) Find $\dfrac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y=x(x+3)^5\ln(x)$

(b) $y=\dfrac{(e^x+x^4)^3}{\sec(5x)+1}$

(c) $\tan(x+y)=x+y^2$

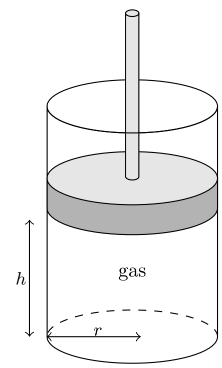
(d) $x^y=y^{\sin(x)}$

6. (4 points) Use logarithmic differentiation to find the derivative of the function

$$y=\frac{\sec^3(x)\sqrt[3]{x+2}}{(2x+1)^7x^x}$$

7. (4 points) Let $f(x)=xe^{3x}$. Find the interval(s) on which f is concave down.

8. (5 points) A sample of gas is placed in a cylindrical chamber held at constant temperature, and a piston is seated on top. The piston is lowered at a constant rate of 3 cm/s. At the moment when the piston is 6 cm from the base of the chamber, the pressure in the chamber is 100 kPa. What is the rate of change of the pressure in the chamber at this moment?
(You are given Boyle's Law: $PV=k$, where P is the pressure of the gas, V is the volume of the gas, and k is a constant.)



9. (4 points) Show that the equation $10x-\sin(x)-5=0$ has *exactly* one real root.

10. (4 points) Find the absolute (global) extrema of the function $f(x)=x-6\sqrt[3]{2x-3}$ on the interval $[-12,15]$.

11. (10 points) Given

$$f(x)=\frac{(x-1)^2(2x+1)}{x^3}$$
$$f'(x)=\frac{3(x^2-1)}{x^4}$$
$$f''(x)=\frac{6(2-x^2)}{x^5}$$

with $f(\sqrt{2})\approx 0.23$ and $f(-\sqrt{2})\approx 3.77$, find:

(a) domain of f,

(b) x and y intercepts,

(c) vertical and horizontal asymptotes,

(d) intervals on which f is increasing or decreasing,

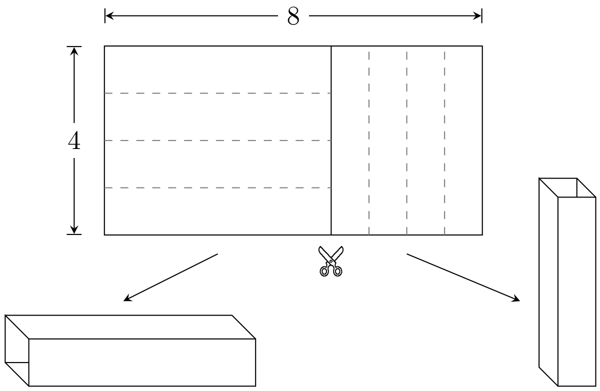
(e) local extrema,

(f) intervals on which f is concave upward or downward,

(g) inflection points.

Sketch the graph of f. Label all intercepts, asymptotes, extrema and inflection points.

12. (6 points) A rectangular piece of paper measuring 4 inches by 8 inches is cut into two pieces. Each piece is then folded along the dashed lines (see below) to make two rectangular prisms with square open ends.
(Note: in the extreme cases, it is also allowed to not cut the paper and only make one of the prisms.)



(a) Where should the cutting line be in order to **minimize** the total volume of the rectangular prisms?

(b) Where should the cutting line be in order to **maximize** the total volume of the rectangular prisms?

13. (4 points) Find $f(t)$ if $f''(t) = 3\sqrt{t} + 2$, $f(1) = 0$ and $f'(1) = 3$.

14. (3 points) Use **differentiation** to verify that the following formula is correct.

$$\int \cos^3(x)\,dx = \frac{1}{3}\left[2 + \cos^2(x)\right]\sin(x) + C.$$

15. (2 points) Express the definite integral $\int_1^3 \frac{\ln(x)}{x}\,dx$ as a limit of Riemann sums. Do not evaluate the limit.

16. (12 points) Evaluate each of the following integrals

(a) $\int \left(\sqrt{\frac{6}{x}} + \sqrt{6}x + 6^x + e^6\right) dx$

(b) $\int_1^2 \frac{x^2+4x+3}{x^2+x}\,dx$

(c) $\int \cos(x)\left[\tan(x) + \sec^3(x)\right] dx$

(d) $\int_0^5 \left(\,|x-2| + 1\,\right) dx$

17. (2 points) Let $F(x) = \int_2^{e^x} \frac{t}{\ln(t)}\,dt$. Find the derivative of this function, $F'(x)$.

18. (2 points) Find the quadratic polynomial

$$p(x) = ax^2 + bx + c$$

that satisfies both

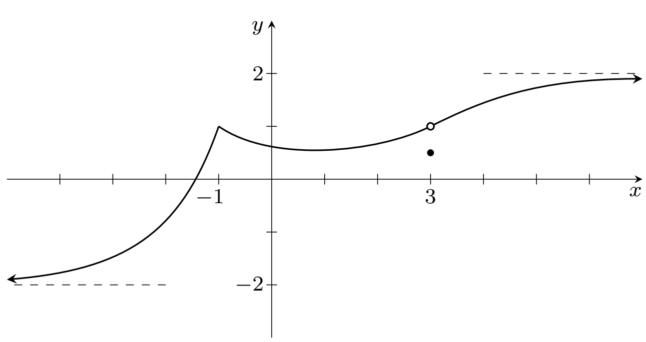
$$\lim_{x\to\infty} \frac{p(x)}{x^2 - x - 6} = 2 \quad \text{and} \quad \lim_{x\to 3} \frac{p(x)}{x^2 - x - 6} = 1.$$

ANSWERS

1. (a) $\frac{1}{18}$ (b) $-\infty$ (c) $-\frac{3}{4}$ (d) $\frac{3}{5}$ (e) 0

2. $(a, b) = (9, 3)$ or $(-26, -4)$

3.



Other answers are possible. For example, f could have a vertical tangent at -1 instead of a corner.

4. (a) Using the limit definition,

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{\sqrt{4-3(x+h)}} - \frac{2}{\sqrt{4-3x}}}{h} = \frac{3}{(4-3x)^{3/2}}$$

(b) Using the power rule and the chain rule,

$$f'(x) = 2\cdot(-1/2)(4-3x)^{-3/2}\cdot(-3) = \frac{3}{(4-3x)^{3/2}}$$

(c) $y = 3x - 1$

5. Warning: some of the answers below have been simplfied and/or reformatted.

(a) $y' = (x+3)^5 \ln x + 5x(x+3)^4 \ln x + (x+3)^5$

(b) $y' = \dfrac{3(e^x + x^4)^2(e^x + 4x^3)}{1 + \sec 5x} - \dfrac{5(e^x + x^4)^3 \sec 5x \tan 5x}{(1 + \sec 5x)^2}$

(c) $y' = \dfrac{1 - \sec^2(x+y)}{\sec^2(x+y) - 2y}$

(d) $y' = \dfrac{\cos(x)\ln(y) - \frac{y}{x}}{\ln(x) - \frac{\sin(x)}{y}}$

6. $y' = \dfrac{\sec^3(x)\,\sqrt[3]{x+2}}{(2x+1)^7\,x^x}\left[3\tan x + \dfrac{1}{3(x+2)} - \dfrac{14}{2x+1} - \ln x - 1\right]$

7. $(-\infty, -\frac{2}{3})$

8. The volume of a cylinder is $V = \pi r^2 h$, therefore

$$PV = P \cdot \pi r^2 h = k \qquad \Rightarrow \qquad P \cdot h = \frac{k}{\pi r^2}$$

By noting that the right-hand side ($\frac{k}{\pi r^2}$) is a constant, taking a derivative and using the product rule we find

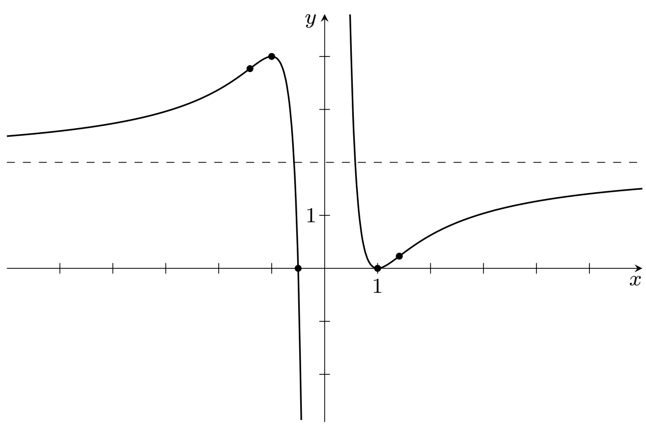
$$\frac{dP}{dt} \cdot h + P \cdot \frac{dh}{dt} = 0$$

which leads to the answer $\frac{dP}{dt} = 50$ kPa/s.

9. IVT: $f(0) = -5 < 0$ and $f(\pi) = 10\pi - 5 > 0$
Therefore, by IVT, there is at least one root in $(0, \pi)$.
Rolle's: $f'(x) = 10 - \cos(x)$ which is never equal to 0 since $-1 \leq \cos(x) \leq 1$. Therefore, there is at most one root to the equation (otherwise contradicting Rolle's theorem).

10. Absolute maximum at $x = -\frac{5}{2}$ (with value of $y = \frac{19}{2}$).
Absolute minimum at $x = \frac{11}{2}$ (with value of $y = -\frac{13}{2}$).
(Note that the function has **three** critical numbers: $x = -\frac{5}{2}$, $x = \frac{3}{2}$ and $x = \frac{11}{2}$.)

11. x-intercepts: $-\frac{1}{2}$ and 1, y-intercept: none,
VA: $x = 0$, HA: $y = 2$,
local min.: $(1, 0)$, local max.: $(-1, 4)$,
inflection points:
$(-\sqrt{2},\ 2 + \frac{5}{4}\sqrt{2}) \approx (-1.4, 3.8)$,
$(\sqrt{2},\ 2 - \frac{5}{4}\sqrt{2}) \approx (1.4, 0.2)$



12. (a) Cut 6 inches from the left side of the paper.
(b) Use the whole sheet of paper to make a *vertical* prism (as on the right side of the figure).

13. $f(t) = \frac{4}{5}t^{5/2} + t^2 - t - \frac{4}{5}$

14.
$$\frac{d}{dx}\left[\tfrac{1}{3}(2 + \cos^2 x)\sin x\right]$$
$$= \tfrac{1}{3}\left[(-2\cos x \sin x)\sin x + (2 + \cos^2 x)\cos x\right]$$
$$= \tfrac{1}{3}\left[2\cos x - 2\cos x \sin^2 x + \cos^3 x\right]$$
$$= \tfrac{1}{3}\left[2\cos x(1 - \sin^2 x) + \cos^3 x\right]$$
$$= \tfrac{1}{3}\left[2\cos x \cdot \cos^2 x + \cos^3 x\right] = \cos^3 x$$

15. Using equal subintervals of $[1, 3]$ and taking the sample points to be right endpoints, we have

$$\Delta x = \frac{2}{n}, \qquad x_i = 1 + \frac{2i}{n} \quad \text{for } i = 1,\, 2,\, \ldots,\, n$$

and so the given integral equals

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\ln(1 + 2i/n)}{1 + 2i/n} \cdot \frac{2}{n}$$

(Other answers are possible.)

16. (a) $2\sqrt{6x} + \frac{1}{2}\sqrt{6}\,x^2 + \frac{6^x}{\ln 6} + e^6 x + C$

(b) $1 + 3\ln(2)$

(c) $-\cos(x) + \tan(x) + C$

(d) $\frac{23}{2}$

17. $F'(x) = \dfrac{e^{2x}}{x}$

18. $p(x) = 2x^2 - 7x + 3$