Question 1: (*8 pts*) Solve each of the following systems or show that the system has no solution. Give parametric solutions where applicable.

a) $\begin{cases} 2x + 6y - 2z = -10 \\ 2x + 5y + 3z = 16 \\ 3x - 7y - 4z = -4 \end{cases}$ **b)** $\begin{cases} 2x_1 + 4x_2 - 2x_3 + 14x_4 = -6 \\ -3x_1 - 6x_2 + 5x_3 - 27x_4 = 16 \\ 2x_1 + 4x_2 - 6x_3 + 26x_4 = -11 \end{cases}$

Question 2: (*4 pts*)

Find the value(s) of h and k so that the system $\begin{cases} x + 3y - 2z = 2 \\ 2x + 5y + z = -3 \\ -3x - 4y + hz = k \end{cases}$ has:

a) a unique solution.
b) no solution.
c) infinitely many solutions.

Question 3: (*6 pts*) A company makes 3 kinds of snacks, using almonds and raisins. The Fruity snack contains 100g of almonds and 300g of raisins. The Nutty snack contains 300g of almonds and 100g of raisins. The Variety snack contains 200g of almonds and 200g of raisins. There are currently 900g of almonds and 700g of raisins available, and we want to determine how many of each kind of snacks can be made so that all the ingredients are completely used.

a) Define variables and set up a linear system in order to solve the problem.
b) Find all the realistic solutions to the problem.

Question 4: (*7 pts*) Given the matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -3 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & a^2 - 2 & 6 \\ -1 & 2 & b \\ 6 & b^2 & 3 \end{bmatrix}$, find (if possible):

a) $(AB)^{-1}$
b) CA^2
c) a matrix X that satisfies $(2A - 3I)\,X = C^T$
d) all the possible values (if any) of a and b so that D is symmetric.

Question 5: (*8 pts*) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 \\ 1 & 3 & 4 & 4 \\ 2 & 3 & 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

a) Find A^{-1} using row reduction.
b) Solve $AX = B$ using A^{-1}.
c) Find $\det(A)$.
d) Use A^{-1} to find $\operatorname{adj}(A)$.

Question 6: (*5 pts*) Use Cramer's Rule to solve ***only*** for y in the system $\begin{cases} 3x - 4y + 5z = 8 \\ 7x + 3y - 2z = -2 \\ 2x - 8y + z = 5 \end{cases}$

Question 7: (*6 pts*) A simple economy has two industries: sugar and spice. In order to produce \$1 of sugar, it takes 40¢ of spice. In order to produce \$1 of spice it takes \$1 of sugar and 50¢ of spice.

a) Which industries, if any, are profitable? Justify your answer.
b) Determine the production necessary to satisfy an external demand of \$23 000 of sugar and \$14 000 of spice.
c) How much of each product is used internally?
d) Is the economy productive? Justify your answer.

Question 8: (*9 pts*) Let $A = \begin{bmatrix} 1 & 4 & 3 & -1 & -4 & -4 \\ 2 & 6 & 5 & -1 & -8 & -7 \\ -5 & 2 & -4 & 4 & 20 & 19 \\ 3 & 2 & 4 & 1 & -12 & -8 \end{bmatrix}$, where $\vec{a_1}$, $\vec{a_2}$, $\vec{a_3}$, $\vec{a_4}$, $\vec{a_5}$ and $\vec{a_6}$ represent the columns of A. A reduces to $R = \begin{bmatrix} 1 & 0 & 1 & 0 & -4 & -3 \\ 0 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Choose a basis for Col(A) from the columns of A.
b) Express each column of A that is not in your basis as a linear combination of your basic vectors.
c) Is $\{\vec{a_3}, \vec{a_4}, \vec{a_5}\}$ linearly independent? Justify your answer.
d) Write (if possible) $\vec{a_1}$ as a linear combination of $\vec{a_4}$ and $\vec{a_6}$.
e) Give a basis for Col(A) that contains $\vec{a_6}$.
f) Give a basis for Nul(A).

Question 9: (*4 pts*) Find both ***unit*** vectors that are orthogonal to $\vec{u} = (2, 3, -2)$ and $\vec{v} = (6, 6, -4)$.

Question 10: (*4 pts*) Let $S = \{(x, y, z) \mid 4x - z = 0,\ 2x - 5y - 6z = 0\}$. Is S a subspace of $\mathbb{R}^3$? If so, find a basis for S. If not, provide a counter-example to show that one of the closure properties fails.

Question 11: (*7 pts*) Given the points $A = (2, -1, 6)$, $B = (4, 2, 7)$, $C = (-1, 3, 5)$ and $D = (5, 12, 7)$:

a) Find the distance from A to B.
b) Write a vector equation for the line parallel to $\overrightarrow{AB}$ passing through C.
c) Is the point D on the line? Justify your answer.
d) Find an equation (in general form) of the plane spanned by $\overrightarrow{AB}$ and $\overrightarrow{CD}$.
e) Is that plane a subspace of $\mathbb{R}^3$?

Question 12: (*4 pts*) Determine if the following sets are linearly independent or linearly dependent. Justify your answer.

a) $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ **b)** $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ d) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 12 \end{bmatrix} \right\}$

Question 13: (*3 pts*) Let A be a 7×4 matrix.

a) Is it possible for $A\mathbf{x} = \mathbf{0}$ to have a unique solution? Justify your answer.
b) What is the minimum possible value for the nullity of A^T?
c) If rank(A)=2, how many parameters are in the solution of $A^T\mathbf{x} = \mathbf{0}$? Justify your answer.

Question 14: (*5 pts*) Suppose that A, B, C and D are 5×5 matrices such that $\det(A) = 7$, $\det(B) = -4$, $\det(C) = 3$ and D is not invertible. Find:

a) $\det(AB)$
b) $\det(2C^{-1})$
c) $\det(B^T A C^2)$
d) $\det(ADB)$
e) nullity(C)

Question 15: (*6 pts*) Solve the following using the graphical method:

$$\text{Maximize } z = x + 4y \text{, subject to the constraints } \begin{cases} x - y & \leqslant & 15 \\ 2x + 5y & \leqslant & 100 \\ 3x + 2y & \geqslant & 40 \\ x & \geqslant & 0 \\ y & \geqslant & 5 \end{cases}$$

Question 16: (*8 pts*) Jackets Plus produces three different jackets, each made from polyester, wool and cotton. Jacket A is made using 30 units of polyester, 20 units of wool, and 20 units of cotton. Jacket B is made using 28 units of polyester, 10 units of wool and 12 units of cotton. Jacket C is made from 25 units of polyester, 20 units of wool, and 10 units of cotton. The manufacturer has on hand 5000 units of polyester, 5000 units of wool, and 3000 units of cotton. Jacket A type sells for \$80 per unit, jacket B sells for \$60 per unit and jacket C sells for \$60 per unit. We want to know how many of each type of jacket should be manufactured and sold to maximize revenue.

a) Set up a linear program to solve this problem.
b) What is the maximum revenue? How many of each type of jackets are manufactured and sold?
c) How much of each resource is left over when the revenue is maximum?

Question 17: (*6 pts*)

Use the simplex algorithm to show that there is no minimum value for C in the problem below. Also find a feasible solution for which $C \leqslant -18\,000$.

$$\text{Minimize } C = -8x - 6y + 2z \text{ subject to } \begin{cases} 3x & + & 9y & - & 10z & \leqslant & 20 \\ -2x & - & 8y & + & 6z & \leqslant & 18 \\ x & + & 2y & - & 2z & \leqslant & 5 \\ x \geqslant 0 & , & y \geqslant 0 & , & z \geqslant 0 & & \end{cases}$$

ANSWERS:

1.a) (3,-1,5) **b)** No solution. **2.a)** $h \neq -19;\ k \in \mathbb{R}$ **b)** $h = -19;\ k \neq 29$ **c)** $h = -19;\ k = 29$

3.a) $x = \#$ Fruity snacks; $y = \#$ Nutty snacks; $z = \#$ Variety snacks $\begin{cases} 100x + 300y + 200z &= 900 \\ 300x + 100y + 200z &= 700 \end{cases}$

b) (1,2,1) and (0,1,3). **4.a)** $\begin{bmatrix} 19/7 & -10/7 \\ -10/7 & 6/7 \end{bmatrix}$ **b)** $\begin{bmatrix} 14 & 49 \\ 14 & 56 \\ -21 & -63 \end{bmatrix}$ **c)** $\begin{bmatrix} -1/5 & 12/5 & 21/5 \\ 1/5 & -2/5 & -6/5 \end{bmatrix}$

d) $a = 1$ or -1, $b = 0$ or 1. **5.a)** $\begin{bmatrix} 4 & -1 & 0 & 0 \\ -8 & 3 & -1 & 0 \\ 24 & -9 & 5 & -1 \\ -19 & 7 & -4 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 4 \\ -9 \\ 29 \\ -23 \end{bmatrix}$ **c)** 1

d) $\begin{bmatrix} 4 & -1 & 0 & 0 \\ -8 & 3 & -1 & 0 \\ 24 & -9 & 5 & -1 \\ -19 & 7 & -4 & 1 \end{bmatrix}$ **6.** $y = \dfrac{-131}{305}$ **7.a)** Sugar is profitable, Spice is not profitable.

b) \$255 000 of Sugar and \$232 000 of Spice. **c)** \$232 000 of Sugar and \$218 000 of Spice. **d)** Yes.

8.a) $\left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_4}\right\}$ **b)** $\overrightarrow{a_3} = \overrightarrow{a_1} + \frac{1}{2}\overrightarrow{a_2};\ \overrightarrow{a_5} = -4\overrightarrow{a_1};\ \overrightarrow{a_6} = -3\overrightarrow{a_1} + \overrightarrow{a_4}.$ **c)** Yes.

d) $\overrightarrow{a_1} = \frac{1}{3}\overrightarrow{a_4} - \frac{1}{3}\overrightarrow{a_6}$ **e)** $\left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_6}\right\}$ **f)** $\left\{ \begin{bmatrix} -1 \\ -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

9. $\begin{bmatrix} 0 \\ 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}$ **10.** S is a subspace, $S = \text{span}\left\{ \begin{bmatrix} 1/4 \\ -11/10 \\ 1 \end{bmatrix} \right\}$ **11.a)** $\sqrt{14}$

b) $\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ **c)** Not on the line **d)** $3x - 2y = 0$ **e)** Yes. **12.a)** LI

b) LD **c)** LD **d)** LI **13.a)** Yes. **b)** 3 **c)** 5 **14.a)** -28 **b)** 32/3

c) -252 **d)** 0 **e)** 0 **15.** Maximum: $z = 80$ at $(0, 20)$ **16.a)** $x = \#$ jacket A;

$y = \#$ jacket B; $z = \#$ jacket C; Maximize $P = 80x + 60y + 60z$ subject to $\begin{cases} 30x + 28y + 25z &\leqslant 5000 \\ 20x + 10y + 20z &\leqslant 5000 \\ 20x + 12y + 10z &\leqslant 3000 \end{cases}$

b) Maximum Profit = \$13 000 with 125 jackets A, 0 jackets B and 50 jackets C.

c) 0 of polyester, 1500 of wool and 0 of cotton.

17. $z = -18\,236$ when $x = 2033$, $y = 1000$ and $z = 2014$ ($s_1 = 5061$, $s_2 = 0$ and $s_3 = 0$).