1. (6 points) Given the graph of f below, evaluate the following expressions. If appropriate use ∞, $-\infty$, or "does not exist" where appropriate.

(a) $\lim_{x\to 5^-} f(x)$

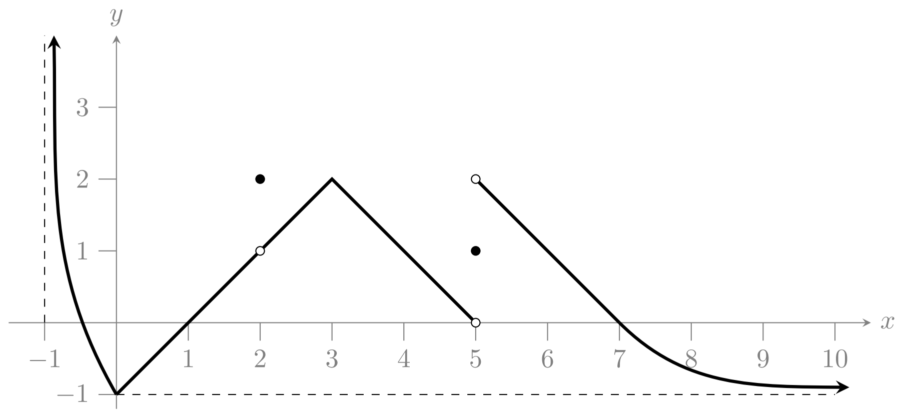
(b) $\lim_{x\to 5^+} f(x)$

(c) $f(f(5))$

(d) $\lim_{x\to -1^+} f(x)$

(e) $\lim_{h\to 0} \dfrac{f(5/4+h)-f(5/4)}{h}$

(f) $f''(4)$



2. (10 points) Evaluate the following limits.

(a) $\lim_{x\to 1} \dfrac{3x^2-x-2}{x^2-x}$

(b) $\lim_{x\to\infty} \left(\dfrac{1+x}{6+5x^2}\right)\left(\dfrac{5+7x^3}{2-5x^2}\right)$

(c) $\lim_{\theta\to 0} \dfrac{\tan(6\theta)}{\sin(8\theta)}$

(d) $\lim_{x\to\infty} \left(\sqrt{4x^2+3x-2}-3x\right)$

(e) $\lim_{x\to 1^+} \dfrac{x+1}{x-|2-3x|}$

3. (4 points) Let

$$f(x) = \begin{cases} \dfrac{1}{k+1-x} & \text{if } x \le 3, \\ \sqrt{\dfrac{x^2-5x+6}{k(x-3)}} & \text{if } x > 3. \end{cases}$$

Find all values of k that make the function $f(x)$ continuous at $x=3$.

4. (3 points) Find an equation of the normal line to the curve $y = \dfrac{x^2}{x-2}$ at the point with x-coordinate equal to 3.

5. (4 points) Find the derivative of $f(x) = \dfrac{1}{2x+1}$ using the limit definition of the derivative.

6. (15 points) Find $\dfrac{dy}{dx}$ for each of the following.

(a) $y = \dfrac{\sqrt[3]{x}}{2} + \dfrac{2}{x+1} - 3^x + \cos(e^2)$

(b) $y = \dfrac{x}{x+1} + \ln\left(\dfrac{2}{x}\right)$

(c) $y = \csc^2(3x^2) + \ln(4-x) + xe^{3x^2}$

(d) $y = (x-1)(2+x)^{2x}$

(e) $\sin(x-y) = xy$

7. (2 points) A particle moves along a straight line with its position at time t given by $s(t) = t^{2/3}(20-t)$. What is the distance travelled by the particle during the time interval $[1, 27]$?

8. (4 points) Let $f(x) = e^x(x^3 - 3x^2 + 6x + 2)$.

(a) Justify that $f(x)$ has a root in the interval $(-1, 0)$.

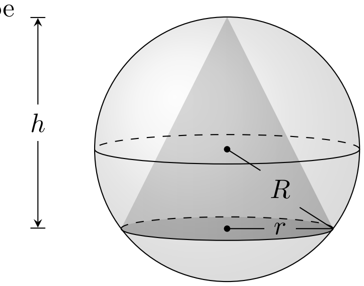
(b) Justify that $f(x)$ has only one root in the interval $(-1, 0)$.

9. (5 points) Find all the points on the graph of the equation $x^4 + y^4 + 2 = 4xy^3$ at which the tangent line is horizontal.

10. (5 points) A plane, flying in a straight line at a constant altitude of 4 km, passes directly over a telescope tracking it. At a certain moment the angle between the telescope's line of sight and the ground is $\pi/3$ and is decreasing at a rate of $1/2$ radians per minute. How fast is the plane travelling at that moment?

11. (4 points) Find the absolute extrema of $f(x) = 15 + 12x - x^3$ on $[1, 4]$.

12. (5 points) Find the height of the right circular cone of largest volume that can be inscribed in a sphere of radius R. ($V = \frac{1}{3}\pi r^2 h$)



13. (10 points) Given $f(x) = \dfrac{6 - 2e^x}{e^x + 1}$, $f'(x) = -\dfrac{8e^x}{(e^x+1)^2}$, $f''(x) = \dfrac{8e^x(e^x-1)}{(e^x+1)^3}$,

find (if any):

(a) domain of f,

(b) x and y intercept(s),

(c) equations of all asymptotes,

(d) intervals on which f is increasing or decreasing,

(e) local (relative) extrema,

(f) intervals of upward or downward concavity,

(g) inflection points(s).

(h) On the next page, sketch the graph of f. Label all intercepts, asymptotes, extrema, and points of inflection.

14. (2 points) Given that $f'(x) = 3\sin(x) + \dfrac{1}{\pi}$ and $f(3\pi/4) = 0$, find $f(x)$.

15. (5 points) Compute the definite integral $\int_1^4 (x^2 - x + 1)\,dx$ as a limit of Riemann sums.

Note that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

16. (2 points) Find a number b and a function f such that

$$\int_2^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{3 + \frac{4i}{n}} \left(\frac{4}{n}\right).$$

17. (3 points) Evaluate $\int_{-2}^{2} \left(|x| + \sqrt{4 - x^2}\right) dx$ by interpreting it in terms of areas.

18. (9 points) Evaluate the following integrals.

(a) $\int \frac{(\sqrt{x} - 1)^2}{x}\,dx$

(b) $\int \left(e^{x+1} + \frac{\sec x \tan x}{2} + 3\sec^2 x\right) dx$

(c) $\int_1^e \left(\frac{2}{t} + \frac{1}{e}\right) dt$

19. (2 points) Given $F(x) = \int_x^{2x} \left(\frac{\sin t}{t}\right) dt$ find $F'(x)$.

Exam Solutions

1. (a) 0 (b) 2 (c) 0 (d) ∞ (e) 1 (f) 0

2. (a) $\lim_{x\to 1} \frac{3x+2}{x} = 5$

   (b) $\lim_{x\to\infty} \frac{7x^4}{-25x^4} = -\frac{7}{25}$

   (c) $\lim_{\theta\to 0} \frac{\sin(6\theta)}{6\theta} \cdot \frac{8\theta}{\sin(8\theta)} \cdot \frac{3}{4\cos(6\theta)} = \frac{3}{4}$

   (d) $\lim_{x\to\infty} (2x - 3x) = -\infty$

   (e) $\lim_{x\to 1^+} \frac{x+1}{x + (2-3x)} = \lim_{x\to 1^+} = \frac{x+1}{-2(x-1)} = -\infty$

3. $\frac{1}{k-2} = \sqrt{\frac{1}{k}} \Rightarrow$ ~~$k = 1$~~ (extraneous), $k = 4$

4. $\frac{dy}{dx} = \frac{x^2 - 4x}{(x-2)^2} \Rightarrow -\left.\frac{dx}{dy}\right|_{x=3} = \frac{(3-2)^2}{4(3) - (3)^2} = \frac{1}{3}$

   $y(3) = 9$

   $y = \frac{1}{3}(x - 3) + 9 \Rightarrow y = \frac{1}{3}x + 8$

5. $\lim_{h\to 0} \frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h} = \frac{-2}{(2x+1)^2}$

6. (a) $\frac{dy}{dx} = \frac{1}{6\sqrt[3]{x^2}} - \frac{2}{(x+1)^2} - \ln(3)3^x$

(b) $\frac{dy}{dx} = \dfrac{1}{(x+1)^2} - \dfrac{1}{x}$

(c) $\frac{dy}{dx} = -12x\csc^2(3x^2)\cot(3x^2) - \dfrac{1}{4-x} + e^{3x^2} + 6x^2e^{3x^2}$

(d) $\frac{dy}{dx} = \left(\frac{1}{x-1} + 2\ln(2+x) + \frac{2x}{2+x}\right)(x-1)(2+x)^{2x}$

(e) $\frac{dy}{dx} = \dfrac{\cos(x-y)-y}{x+\cos(x-y)}$

7. $s'(t) = \dfrac{40-5t}{3\sqrt[3]{t}} \Rightarrow$ critical points: $t = 0, 8$

$s'(t) > 0$ for $1 < t < 8$ and $s'(t) < 0$ for $8 < t < 27 \Rightarrow$ distance $= (s(8) - s(1)) + (s(8) - s(27)) = 128$ units.

8. (a) Note $f(x)$ is continuous on $\mathbb{R}$, and $f(-1) = \frac{-8}{e} < 0$, $f(0) = 2 > 0$. Conclude by IVT.

(b) Suppose $f(x)$ has a second root in $(0,-1)$. $f(x)$ is differentiable on $\mathbb{R}$, so Rolle's Theorem would assure that $f'(x) = 0$ at some point in $(0,-1)$ between these roots. But $f'(x) = e^x(x^3+8) > 0$ on $(0,-1)$. So there can be no second root on $(0,-1)$.

9. $4x^3 + 4y^3y' = 4y^3 + 12xy^2y' \Rightarrow y' = \dfrac{4(x^3-y^3)}{4y^2(3x-y)}$

$y' = 0 \Rightarrow x = y \Rightarrow 2y^4 + 2 = 4y^2 \Rightarrow y = \pm 1$ Points: $(1,1)$, $(-1,-1)$

10. $\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{4}{x}\right) \Rightarrow \sec^2(\theta)\frac{d\theta}{dt} = -\frac{4}{x^2}\frac{dx}{dt}$

$\frac{d\theta}{dt} = -\frac{1}{2}, \theta = \frac{\pi}{3}$

$\tan(\pi/3) = \frac{4}{x} \Rightarrow x = \frac{4}{\sqrt{3}}$

$\frac{dx}{dt} = 4 \cdot \frac{1}{2} \cdot \frac{16/3}{4} = \frac{8}{3}$km/min

11. $f'(x) = 12 - 3x^2 \Rightarrow$ critical points: $x = \pm 2$.

$f(1) = 26$, $f(2) = 31$, $f(4) = -1 \Rightarrow$ Local max: $(2, 31)$, Local min: $(4, -1)$.

12. We have $r^2 = R^2 - (h-R)^2 = 2Rh - h^2$, so

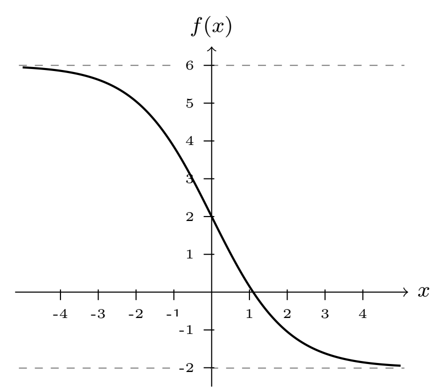
$V(h) = \frac{\pi}{3}(2Rh^2 - h^3)$

$V'(h) = \frac{\pi}{3}\left(4Rh - 3h^2\right) = \pi h\left(\frac{4}{3}R - h\right)$

Critical points: ~~$h = 0$~~, $h = \frac{4}{3}R$

Check: $V''(\frac{4}{3}R) = \frac{\pi}{3}(4R - 6(\frac{4}{3}R)) < 0$ so $h = \frac{4}{3}R$ yields the maximal volume.

13. Domain: $\mathbb{R}$
$x-$int: $(\ln(3), 0)$,
$y-$int: $(0, 2)$
V.A.: None
H.A.:$y = 6$ on the left, $y = -2$ on the right.
Decrease: $\mathbb{R}$, Critical points: None.
Possible inflection points: $x = 0$.
Concave up: $(0,\infty)$, Concave down: $(-\infty, 0)$,
Inflection point at $(0, 2)$.



14. $f(x) = -3\cos(x) + \frac{x}{\pi} + C$

$-3\cos(3\pi/4) + \frac{3\pi}{4\pi} + C = 0 \Rightarrow C = -\frac{3+6\sqrt{2}}{4}$

$f(x) = -3\cos(x) + \frac{x}{\pi} - \frac{3+6\sqrt{2}}{4}$

15.

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(1+\frac{3}{n}i\right)^2-\left(1+\frac{3}{n}i\right)+1\right)\frac{3}{n} &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{3}{n}i+\frac{9}{n^2}i^2\right)\frac{3}{n}\\ &= \lim_{n\to\infty}\left(\frac{3n}{n}+\frac{9n(n+1)}{2n^2}+\frac{27n(n+1)(2n+1)}{6n^3}\right)\\ &= 3+\frac{9}{2}+9=\frac{33}{2}\end{aligned}$$

16. $b=6,\ f(x)=\sqrt{1+x}$

17. $\int_{-2}^{2}|x|dx+\int_{-2}^{2}\sqrt{4-x^2}dx=4+2\pi$

18. (a) $\int \frac{x-2x^{1/2}+1}{x}dx=\int 1-2x^{-1/2}+\frac{1}{x}dx=x-4\sqrt{x}+\ln|x|+C$

(b) $e^{x+1}+\frac{\sec(x)}{2}+3\tan(x)+C$

(c) $(2\ln|t|+\frac{t}{e})\big|_1^e=(2\ln(e)+\frac{e}{e})-(2\ln|1|+\frac{1}{e})=3-\frac{1}{e}$

19. $\frac{\sin(2x)-\sin(x)}{x}$