(Marks)

(5) 1. Let $g(x) = \int_0^x \arctan\left(\frac{t^2}{2}\right)\, dt$.

(a) Find the Maclaurin series for $g(x)$.

(b) Estimate $g(0.5)$ accurate to within $\pm 5.0 \times 10^{-7}$

(5) 2. Let $f(x) = \sqrt[3]{x}$

(a) Find the Taylor series for $f(x)$ centered at $x = 27$; express your answer in Σ notation.

(b) Find the third degree Taylor polynomial $T_3(x)$ for $f(x)$ centered at $x = 27$.

(c) Use Taylor's inequality to find an upper bound for the error when approximating $f(25)$ by $T_3(x)$.

(8) 3. Consider the following polar curves: $r_1 = -\cos\theta$ and $r_2 = 1 + \cos\theta$.

(a) Sketch the graphs on the same axes.

(b) Find all points of intersection (in Cartesian coordinates).

(c) Set-up, but do not evaluate, an integral expression to find the area enclosed by both curves.

(d) Set-up, but do not evaluate, an integral expression to find the length of the second curve, r_2.

(8) 4. Given the curve $\mathcal{C}$ with parametric equations $x = t^2 - 2t$, $y = t^2 + 2t$:

(a) Find the x and y-intercepts.

(b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Simplify your answers.

(c) Locate all points where the tangent is horizontal or vertical (identify which is which).

(d) Sketch the curve for $-2 \le t \le 2$, and indicate with an arrow the direction of increasing t values (the orientation).

(e) Set-up, but do not evaluate, an integral expression to determine the area between the curve and the y-axis in the second quadrant.

(f) Set-up, but do not evaluate, an integral expression to determine the arc length of the curve on the interval $-2 \le t \le 2$.

(9) 5. Sketch and name each of the following surfaces in $\mathbb{R}^3$. Show all relevant work.

(a) $r^2 = 4\csc^2\theta$ (b) $2r^2 - \rho^2 = 4$ (c) $z = 4\sqrt{4 - (x-2)^2 - (y-2)^2}$

(10) 6. A particle P moves along a curve $\boldsymbol{r}(t) = \mathrm{e}^t \sin(t)\,\boldsymbol{i} + \sqrt{2}\,\mathrm{e}^t\,\boldsymbol{j} + \mathrm{e}^t \cos(t)\,\boldsymbol{k}$.

(a) Calculate the length of the curve from $t = 0$ to $t = 1$.

(b) Find the unit tangent vector $\boldsymbol{T}(t)$, the unit normal vector $\boldsymbol{N}(t)$,
the curvature $\kappa(t)$, and
the tangential and normal components a_T, a_N of acceleration.
Hint: $(\sin t + \cos t)^2 + (\sin t - \cos t)^2 = 2$.

(6) 7. Let $z = f(x, y) = \mathrm{e}^{(2x-3y)}$.

(a) Find the total differential dz.

(b) Find the tangent plane to the surface $z = f(x, y)$ at $(0, 0)$.

(Marks)

(c) Calculate the linear approximation to $\Delta z = f(Q) - f(P)$, where $P = (0,0)$ and $Q = (0.05, -0.05)$, and so estimate $f(0.05, -0.05)$.

(3) 8. Let $z = f(x,y)$ be a surface, and $f(x,y) = c$ one of its level curves in the xy-plane. Assuming this curve is represented by the vector equation $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$, use the chain rule to show that the gradient of f is always perpendicular to the level curve.

(3) 9. Is the following function continuous at the origin? $f(x,y) = \begin{cases} \dfrac{3x^2 + y^3}{x^2 + 2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Justify your answer.

(4) 10. If $f(u,v,w)$ is a differentiable function, and $g(x,y,z) = f(x-y, y-z, z-x)$, then show that $\dfrac{\partial g}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial g}{\partial z} = 0$.

(4) 11. Given the (level) surface $\mathcal{S}$: $f(x,y,z) = x - y^3 - 2z^2 = 2$ and the point $P(-4,-2,1)$, find:

(a) the directional derivative of f at the point P in the direction of $\boldsymbol{v} = \langle 3, 6, -2\rangle$; and

(b) the parametric equations of the tangent line at P to the curve of intersection of $\mathcal{S}$ and the plane given by $2x - 3y - z = -3$.

(5) 12. Find and classify the critical points of $f(x,y) = 4x - 3x^3 - 2xy^2$.

(6) 13. Find maximum and minimum of $f(x,y,z) = 3x - y - 3z$ subject to two constraints $x + y - z = 0$ and $x^2 + 2z^2 = 1$.

(10) 14. Evaluate

(a) $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2 + 5y}\, dx\, dy$ (b) $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} \arctan\left(\frac{y}{x}\right) dy\, dx$

(4) 15. Let $\mathcal{R}$ be the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$. Set up an appropriate integral to calculate the volume of $\mathcal{R}$.
(You do not have to evaluate the integral.)

(6) 16. Let $\mathcal{S}$ be the region within the cylinder $x^2 + y^2 = 2$ between $z = 0$ and the cone $z = \sqrt{x^2 + y^2}$. Set up the integral of $f(x,y,z) = x^2 + y^2$ over $\mathcal{S}$ using

(a) cylindrical coordinates; (b) spherical coordinates.

(You do not have to evaluate these integrals.)

(4) 17. Evaluate $\displaystyle\iint_{\mathcal{P}} e^{4x-y} dx dy$ where $\mathcal{P}$ is the parallelogram determined by the points $(0,0), (4,1), (3,3), (7,4)$.
Hint: Use the change of variable $x = 4u + 3v$ and $y = u + 3v$.

Answers

1. (a) $g(x) = \int_0^x \arctan\left(\frac{t^2}{2}\right)\, dt = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-1}}{2^{2n-1}(2n-1)(4n-1)}$

   (b) $0.02078683 < g(0.5) < 0.02078683 + 2.8 \times 10^{-7}$

2. (a) $\sqrt[3]{x} = 3 + \frac{1}{27}(x-27) + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{2 \cdot 5 \cdots (3n-4)}{3^{4n-1} n!} (x-27)^n$

   (b) $T_3(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2 + \frac{5}{532441}(x-27)^3$

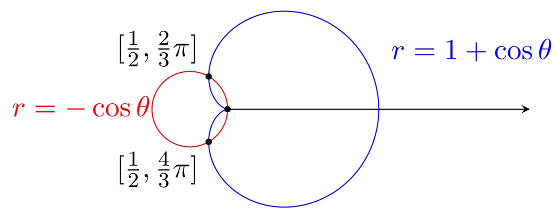
   (c) $|R_3(25)| \leq 4.9 \times 10^{-6}$

3. (a): Graph at right

   (b) Intersections: $(0,0), (-\frac{1}{4}, \frac{\sqrt{3}}{4}), (-\frac{1}{4}, -\frac{\sqrt{3}}{4})$

   (c) $A = 2\left(\frac{1}{2}\int_{\pi/2}^{2\pi/3} \cos^2\theta\, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (1+\cos\theta)^2\, d\theta\right)$

   (d) $s = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\, d\theta$



4. (a) y-intercepts: $(0,0), (0,8)$; x-intercepts: $(0,0), (8,0)$

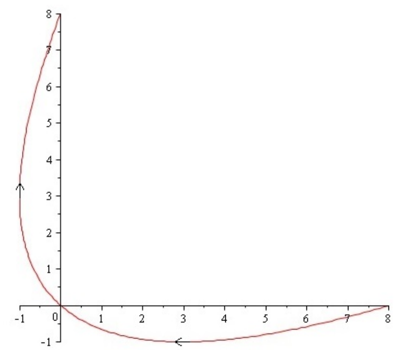
   (b) $\frac{dy}{dx} = \frac{t+1}{t-1}$ and $\frac{d^2y}{dx^2} = \frac{-1}{(t-1)^3}$

   (c) HT at $(3,-1)$ @ $t=-1$; VT at $(-1,3)$ @ $t=1$.

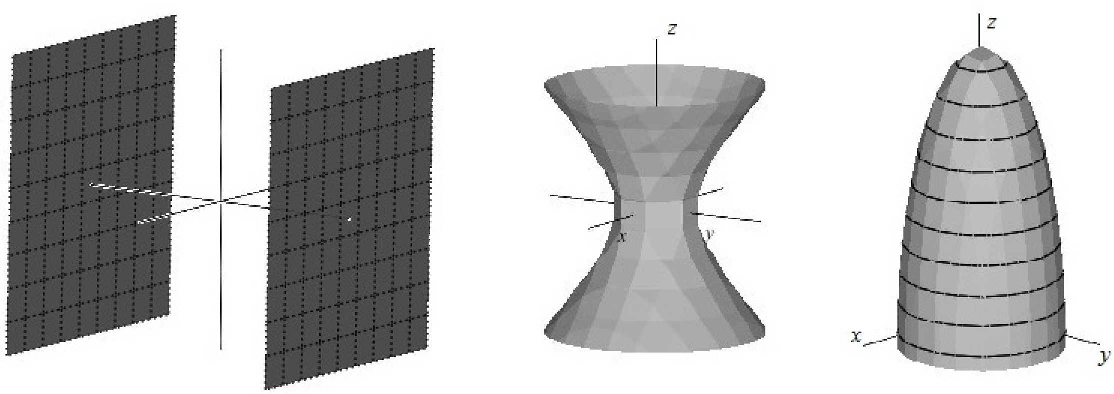
   (d) Graph at right

   (e) $A = \int_0^2 (2t - t^2)(2t+2)\, dt$

   (f) $s = \int_{-2}^{2} \sqrt{(2t-2)^2 + (2t+2)^2}\, dt$



5. Three graphs: (a) a pair of planes (b) a hyperboloid of 1 sheet (c) the top half of an ellipsoid



6. (a) $s = \int_0^1 2\,e^t\, dt = 2(e-1)$

   (b) $\boldsymbol{T}(t) = \frac{1}{2}\langle \sin(t) + \cos(t), \sqrt{2}, \cos(t) - \sin(t)\rangle$; $\boldsymbol{N}(t) = \frac{1}{\sqrt{2}}\langle \cos(t) - \sin(t), 0, -(\sin(t) + \cos(t))\rangle$.

   $\kappa = \frac{1}{2\sqrt{2}} e^{-t}$; $a_T = 2\,e^t$; $a_N = \sqrt{2}e^t$

7. (a) $dz = e^{2x-3y}(2\,dx - 3\,dy)$. (b) Plane: $2x - 3y - z + 1 = 0$

   (c) $\Delta z \sim dz = 0.25$, so $f(0.05, -0.05) \sim 1.25$

8. $\frac{dz}{dt} = 0$ on the level curve (since z is constant there); but (chain rule) $\frac{dz}{dt} = f_x x' + f_y y' = \nabla f \cdot \boldsymbol{r'}$, so $\nabla f \perp \boldsymbol{r'}$. In other words, ∇f is perpendicular to the (tangent to the) level curve.

9. The limit $\lim_{(x,y)\to(0,0)} \frac{3x^2+y^3}{x^2+2y^2}$ does not exist (along the path $x = 0$ it is 0, but along the path $y = 0$ it is 3), so the function cannot be continuous at $(0,0)$.

10. $\frac{\partial g}{\partial x} = f_u - f_w$, $\frac{\partial g}{\partial y} = f_v - f_u$, $\frac{\partial g}{\partial x} = f_w - f_v$, so $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0$.

11. (a) $f_{\boldsymbol{u}}(-4,-2,1) = -61/7$. (b) Equations: $\{x = -4,\ y = -2 - 7t,\ z = 1 + 21t\}$

12. Two saddle points at $(0, \pm\sqrt{2})$; local min at $(-\frac{2}{3}, 0)$; local max at $(\frac{2}{3}, 0)$.

13. Max: $f = \frac{12}{\sqrt{6}} = 2\sqrt{6}$, at $(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$; Min: $f = -\frac{12}{\sqrt{6}} = -2\sqrt{6}$, at $(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}})$.

14. (a) $\displaystyle\int_0^2 \int_0^{x^2} \sqrt{4x^2 + 5y}\, dy\, dx = 152/15$ (b) $\displaystyle\int_0^{\pi/2} \theta\, d\theta \int_0^4 r\, dr = \pi^2$

15. $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{6-r^2}}^{4-r^2} r\, dz\, dr\, d\theta \ = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{6-x^2-y^2}}^{4-x^2-y^2} dz\, dy\, dx$ *(Spherical is a mess!)*

16. (a) $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{r} r^3\, dz\, dr\, d\theta$ (b) $\displaystyle\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}\csc\varphi} \rho^4 \sin^3\varphi\, d\rho\, d\varphi\, d\theta$

17. $\displaystyle\iint_{\mathcal{P}} \mathrm{e}^{4x-y}\ dx\, dy = 9 \int_0^1 \mathrm{e}^{15u}\ du \int_0^1 \mathrm{e}^{9v}\ dv = \frac{1}{15}(\mathrm{e}^{15} - 1)(\mathrm{e}^{9} - 1)$