1. [8 points] Solve each of the following systems or show that the system has no solution. Give parametric solutions where applicable.

(a) $\begin{cases} 2x_1 + x_2 - x_3 = 2 \\ -x_1 + 3x_2 - 10x_3 = 6 \\ 3x_1 - 2x_2 + 9x_3 = -4 \end{cases}$

(b) $\begin{cases} x_1 + 2x_2 - 2x_3 = -3 \\ 3x_1 + 7x_2 - 4x_3 = -7 \\ -2x_1 + 4x_2 + 14x_3 = 20 \end{cases}$

2. [5 points] Given the system below, find a relationship on a, b, and c so that the system has:

(a) A unique solution

(b) No solution

(c) Infintely many solutions

$$\begin{cases} x_1 - 2x_2 + ax_3 = 3 \\ 2x_1 - 5x_2 + bx_3 = 5 \\ -2x_1 - x_2 + cx_3 = -11 \end{cases}$$

3. [5 points] Find an equation for the plane that contains the points $(2, 1, 2)$, $(1, 3, 1)$ and $(1, 8, -14)$ and *sketch* the plane.

4. [5 points] Snack Inc. produces snack bars from oats and dried fruits. Their Plain bar contains 50 grams of oats, their Regular bar contains 40 grams of oats and 10 grams of dried fruits, and their Fruity bar contains 20 grams of oats and 30 grams of dried fruits. One day 5 kilograms of oats and 1.8 kilograms of dried fruits were used. The manager wants to know how many bars of each type were produced that day?

(a) Name variables and set-up a linear system that represents this situation. Do not solve.

(b) Given that the general solution is $(2t - 44, 180 - 3t, t)$, give all values of t which represent realistic solutions.

(c) What specific, realistic solution gives the maximum number of Plain bars?

5. [8 points] Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ -3 & -4 \\ 3 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 0 & -5 \\ 4 & 1 & -6 \end{bmatrix}$, and $D^{-1} = \begin{bmatrix} 5 & 1 \\ 0 & -1 \end{bmatrix}$.

Find the following or state that they are undefined.

(a) BA^T

(b) $(AD)^{-1}$

(c) $3C^2$

(d) $(B^T + C)A$

(e) $\det(A^9)$

6. [4 points] Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 13$, find the following:

(a) $\begin{vmatrix} g & h & i \\ 3g + 5d & 3h + 5e & 3i + 5f \\ 4a & 4b & 4c \end{vmatrix}$

Question 6 continues on the next page.

(b) $\begin{vmatrix} (a+d) & (b+e) & (c+f) \\ g & h & i \\ (a+d-g) & (b+e-h) & (c+f-i) \end{vmatrix}$

7. [2 points] Given that $(I - A)$ is invertible, solve the matrix equation $X = AX + B$ for X, where A, B, and X are $n \times n$ matrices.

8. [2 points] Determine whether the following statements are true or false. Justify your conclusions.

(a) If A is a 3×4 matrix with nullity 1, then the system $AX = B$ must have infinitely many solutions.

(b) If a set of vectors spans $\mathbb{R}^3$, then they form a basis for $\mathbb{R}^3$.

9. [8 points] Let $A = \begin{bmatrix} 2 & 1 & -3 \\ 5 & 0 & 4 \\ 2 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -26 \\ 0 \\ 52 \end{bmatrix}$.

(a) Express $AX = B$ as a system of equations in variables x, y, z.

(b) Find $\operatorname{adj}(A)$.

(c) Find $\det(A)$.

(d) Find A^{-1} using $\operatorname{adj}(A)$.

(e) Solve $AX = B$ using A^{-1}.

(f) Find $(A^T)^{-1}$ and $\operatorname{adj}(A^T)$

10. [4 points] Use Cramer's Rule to solve the system for x_2 only:

$$\begin{cases} 4x_1 + 2x_2 + x_3 + x_4 = 5 \\ x_1 + \quad + 2x_3 + 2x_4 = 0 \\ 2x_1 + x_2 + x_3 + 3x_4 = 0 \\ -x_1 - x_2 - x_3 + 7x_4 = 0 \end{cases}$$

11. [6 points] A simple economy consists of two industries: Energy and Material. To produce \$1 of Energy requires 50¢ of Energy and 60¢ of Material. To produce \$1 of Material requires 80¢ of Energy. There is an external demand for \$600 of Energy and \$400 of Material.

(a) How much should each industry produce to meet the external demand?

(b) Find the internal consumption when the external demand is met.

(c) Which industries, if any, are profitable? Justify your answer.

(d) Is the economy productive? Why or why not?

12. [3 points] Given the points $P(3, -4, -1)$, $Q(7, -3, 2)$, and $R(10, -7, 1)$.

(a) Find the vector $\overrightarrow{PQ}$ and its magnitude.

(b) Find a vector equation for the line L passing through P and Q.

(c) Does the point R lie on the line L? Justify your conclusion.

13. [3 points] Let $\overrightarrow{u_1} = (2, -4, 4)$ and $\overrightarrow{u_2} = (4, -5, -4)$. Find an equation relating x, y, and z for general vector $\overrightarrow{b} = (x, y, z)$ in $Span\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$

14. [6 points] For each set S below, determine if S is a subspace of $\mathbb{R}^3$. If so, find a basis for S. If not, provide a counter-example to show that one of the closure properties fails.

Question 14 continues on the next page.

(a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + 3y - 8z = 2 \right\}$

(b) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rrcrcl} & x & - & 3z & = & 0 \\ \& & 2x & + & y & = & 0 \end{array} \right\}$

15. [9 points] The matrix $A = \begin{bmatrix} 3 & 3 & 0 & 3 & 6 & 14 \\ -2 & -3 & 0 & -3 & -2 & -5 \\ 3 & 6 & 0 & 6 & 0 & 1 \\ 5 & 6 & 0 & 6 & 8 & 13 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

and let $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}, \mathbf{a_6}$ represent the columns of the matrix A.

(a) Find a basis for $\text{Nul}(A)$. What is the *nullity* of A?

(b) Find a basis for the column space of A and express each remaining column as a linear combination of the basis vectors.

(c) Do the columns of A span $\mathbb{R}^4$? Justify.

(d) Determine whether each of the following sets are linearly independent or dependent. Justify.

i. $\{\mathbf{a_1}, \mathbf{a_4}, \mathbf{a_5}\}$ ii $\{\mathbf{a_4}, \mathbf{a_5}, \mathbf{a_6}\}$ iii. $\{\mathbf{a_2}, \mathbf{a_3}\}$

16. [8 points] The Printing Press produces hardcover books and paperbacks at two locations. Location A can produce 100 hardcover books and 300 paperbacks while location B can produce 400 hardcover books and 200 paperbacks per month. The monthly operating cost is \$300 for location A and \$700 for location B. The company can only afford to open one location each month and it needs to produce at least 1000 of each type of book per year. Find the minimum yearly cost to produce at least 1000 of each type of book in one year. Solve using the graphical method.

17. [7 points] Solve using the simplex algorithm. Give the feasible solution including the slack variables.

$$\begin{array}{ll} \text{Minimize} & z = -4x_1 - 9x_2 + x_3 \\ \text{subject to} & 2x_1 + 3x_2 - 2x_3 \le 3 \\ & 7x_1 + 6x_2 + x_3 \le 8 \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0 \end{array}$$

18. [7 points] Use the simplex algorithm to show the following linear program has no maximum, then find a feasible solution with $z = 373$.

$$\begin{array}{ll} \text{Maximize} & z = -x_1 + 9x_2 + 10x_3 \\ \text{subject to} & x_1 - 9x_2 \le 7 \\ & -x_1 + 3x_2 + 2x_3 \le 9 \\ & x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0 \end{array}$$

Answers

1. (a) $x_1 = -t, x_2 = 2 + 3t, x_3 = t$ (b) $x_1 = -5, x_2 = \frac{4}{3}, x_3 = \frac{1}{3}$
2. (a) $c - 5b + 12a \neq 0$ (b) no solution (c) $c - 5b + 12a = 0$

 Question 18 continues on the next page.

3. $5x+3y+z=15$

4. (a) x =number of Plain bars, y =number of Regular bars, z =number of Fruity bars: $\left\{\begin{array}{ccccccc} 50x & + & 40y & + & 20z & = & 5000 \\ & & 10y & + & 30z & = & 1800 \end{array}\right\}$
(b) $\{t \in \mathbb{N} | 22 \leq t \leq 60\}$ (c) 76 Plain bars (when $t=60$)

5. (a) $BA^T = \begin{bmatrix} 19 & 13 \\ -17 & -19 \\ 9 & 3 \end{bmatrix}$ $(b)(\text{AD})^{-1}) = \frac{1}{10}\begin{bmatrix} 19 & -7 \\ 1 & -3 \end{bmatrix}$ (c) undefined (d) undefined (e) 10^9

6. (a) -260 (b) 0

7. $X=(I-A)^{-1}B$

8. (a) True (b) False

9. (a) $\left\{\begin{array}{ccccccc} 2x & + & y & - & 3z & = & -26 \\ 5x & & & + & 4z & = & 0 \\ 2x & - & y & + & z & = & 52 \end{array}\right\}$ (b) $\text{adj}(A) = \begin{bmatrix} 4 & 2 & 4 \\ 3 & 8 & -23 \\ -5 & 4 & -5 \end{bmatrix}$ (c) $\det(A) = 26$ (d) $A^{-1} = \frac{1}{26}\begin{bmatrix} 4 & 2 & 4 \\ 3 & 8 & -23 \\ -5 & 4 & -5 \end{bmatrix}$ (e) $X = \begin{bmatrix} 4 \\ -49 \\ -5 \end{bmatrix}$ (f) $\text{adj}(A^T) = \begin{bmatrix} 4 & 3 & -5 \\ 2 & 8 & 4 \\ 4 & -23 & -5 \end{bmatrix}$ $(A^T)^{-1} = \frac{1}{26}\begin{bmatrix} 4 & 3 & -5 \\ 2 & 8 & 4 \\ 4 & -23 & -5 \end{bmatrix}$

10. $x_2 = \frac{-65}{9}$

11. (a) $C = \begin{bmatrix} 0.5 & 0.8 \\ 0.6 & 0 \end{bmatrix}$, so \$46000 of Energy and \$28000 of Material (b) \$45400 of Energy and \$27600 of Material
(c) Energy is NOT profitable (its column sum is greater than 1), but Material IS profitable (its column sum is less than 1).
(d) Yes, the economy s productive since all entries of $(I-C)^{-1}$ are greater than 0.

12. (a) $\overrightarrow{PQ} = (4,1,3)$ and $\left\|\overrightarrow{PQ}\right\| = \sqrt{26}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ (c) No.

13. (a) $6x+4y+z=0$

14. (a) $\overrightarrow{u} = (1,0,0) \in S$, $\overrightarrow{v} = (2,2,1) \in S$, but $\overrightarrow{u}+\overrightarrow{v} = (3,2,1) \notin S$, so S is not closed under addition. Also, $2\overrightarrow{u} = (2,0,0) \notin S$, so S is not closed under scalar multiplication. (Many solutions possible.) (b) S is a subspace of $\mathbb{R}^3$. Basis for $S: \{(3,-6,1)\}$

15. (a) Basis for $\text{Nul}(A)$: $\left\{\begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$, nullity$=3$ (b) Basis for $\text{Col}(A)$: $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_6}\}$, $\mathbf{a_3} = \mathbf{0a_1}, \mathbf{a_4} = \mathbf{a_2}, \mathbf{a_5} = \mathbf{4a_1} - \mathbf{2a_2}$ (c) No. $Col(A)$ has only 3 basis vectors instead of the necessary 4. (d) i. L.D.: $\mathbf{a_5} - \mathbf{4a_1} + \mathbf{2a_4} = \mathbf{0}$ ii. L.I. iii. L.D.: $1\mathbf{a_3} = \mathbf{0}$

16. At $(0,12), C=\$8400$. At $(12,0), C=\$3600$. At $(10,0), C=\$3000$. At $(2,2), C=\$2000$. At $(0,5), C=\$3500$. So the minimum yearly cost is \$2000.

17. $z=-11, x_1=0, x_2=\frac{19}{15}, x_3=\frac{2}{5}, s_1=0, s_2=0$

18. $z=73+30t, x_1=7+9t, x_2=t, x_3=8+3t, s_1=0, s_2=0$. Therefore, when $t=10$, we have $z=373, x_1=97, x_2=10, x_3=38, s_1=0, s_2=0$.