1. (5 points) Given

$$\begin{aligned} x_1 + 2x_2 + 6x_3 + 4x_4 &= 5 \\ x_2 + 5x_3 + 3x_4 &= 7 \\ x_1 + 3x_2 + 11x_3 + 7x_4 &= 12 \end{aligned}$$

(a) Write the general solution in parametric vector form.

(b) Find a basis for the column space of the coefficient matrix.

2. (8 points) In each part, write a 3×3 matrix A that fits the description or explain why no such matrix exists.

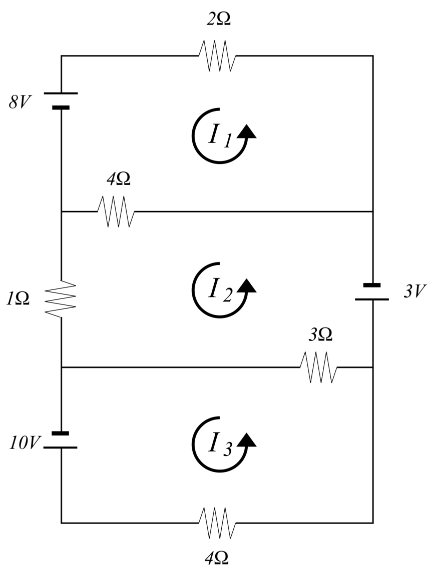
(a) $\dim(\text{Nul}(A)) = 0$

(b) The columns of A form a linearly dependent set, but the rows of A form a linearly independent set.

(c) The null space of A is a plane.

(d) A has rank 1, and $I + A$ is non-invertible.

3. (3 points) Set up an augmented matrix for finding the loop currents of the following electrical circuit. You do not have to solve it.



4. (6 points) Let A, B, and C be 4×4 matrices such that $\det(A) = -2, \det(B) = 3$, and C is non-invertible. Find the value of each of the following:

(a) $\det(-5A^2B^{-1})$

(b) $\det(\text{adj}(B))$

(c) $\det((ABC)^T)$

5. (4 points) Let $A = \begin{bmatrix} 2 & 6 & -1 \\ 1 & 2 & -3 \\ 3 & 7 & -6 \end{bmatrix}$. Find the inverse of A.

6. (4 points) Write an LU Factorization for the matrix $A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 12 & 6 \\ -4 & -22 & -27 \end{bmatrix}$

7. (3 points) Use Cramer's rule to solve the following system for x_2 only.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

8. (5 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation that shifts the first three entries down one spot and brings the negative of the last entry to the top.

For example, $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

(a) Find the standard matrix A of this transformation.

(b) Find A^{87}.

9. (6 points) A non-zero square matrix is said to be **nilpotent of degree 2** if $A^2 = 0$.

(a) Provide an example of a 2×2 matrix that is nilpotent of degree 2.

(b) Show that if A is nilpotent of degree 2, then so is the block matrix $\begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}$.

(c) Suppose A is an $n \times n$ matrix that is nilpotent of degree 2. Is there any non-zero scalar k such that $A + kI$ is nilpotent of degree 2?

10. (10 points) Let $H = \left\{ A \in M_{2\times 2} \colon A \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(a) Find a non-zero matrix in H.

(b) Does H contain the zero matrix? Justify.

(c) Is H closed under addition? Justify.

(d) Is H closed under scalar multiplication? Justify.

(e) Is H a subspace of $M_{2\times 2}$? Justify.

11. (4 points) Find a basis for the vector space $V = \{ \mathbf{p}(t) \in \mathbb{P}_3 \colon \mathbf{p}(-2) = 0, \mathbf{p}(2) = 0 \}$.

12. (10 points) Let $\mathcal{P}$ be the plane $x - 2y - 3z = -4$, and let A be the point $(-3, 1, -2)$.

(a) Find a parametric vector equation for the line through A and perpendicular to $\mathcal{P}$.

(b) Find the point on $\mathcal{P}$ closest to A.

(c) Find an equation of the form $ax + by + cz = d$ of the plane through A and parallel to $\mathcal{P}$.

(d) What is the distance from A to $\mathcal{P}$?

(e) The plane $-4x + 7y + kz = h$ is perpendicular to $\mathcal{P}$ and goes through A. Find k and h.

13. (6 points) Let $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

(a) Find a unit vector $\mathbf{u}$ perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(b) Find the volume the parallelepiped $\mathcal{P}$ formed by $\mathbf{v}$, $\mathbf{w}$, and the vector $\mathbf{u}$ you found in part (a).

(c) Now let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation with standard matrix $A = \begin{bmatrix} 3 & 2 & 9 \\ 0 & -4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$. Find the volume of $T(\mathcal{P})$, that is, the image of $\mathcal{P}$ under T.

14. (4 points) Find a condition on a, b, c so that $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 14 \\ 6 \\ 4 \end{bmatrix} \right\}$

15. (4 points) Let A, B, and C be invertible matrices such that $B^{-1}AB + B^{-1}C = I$.

(a) Solve for A in terms of the other matrices.

(b) Prove that B cannot equal C.

16. (6 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation associated with a standard transformation matrix A.

(a) If $m > n$, find an expression for the maximum possible value of $\dim(\text{Col}(A))$?

(b) If $m > n$, is it possible for T to be one-to-one? Justify.

(c) If $m = 4$ and $n = 6$ and the $\dim(\text{Nul}(A))$ of A is 3, give the dimension of the column space, row space, and null space of A^T.

17. (8 points) Complete each of the following sentences with “must”, “might”, or “cannot”.

(a) If $\mathbf{x} \in \text{Nul}(A)$, then $-2\mathbf{x}$ ________ also be in $\text{Nul}(A)$.

(b) Let $\mathbf{w}$ be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{w}$ ________ be orthogonal to $\mathbf{u} + \mathbf{v}$.

(c) Let $\mathbf{u}$ be parallel to $\mathbf{x}$, and let $\mathbf{v}$ be parallel to $\mathbf{y}$. Then $\mathbf{u} + \mathbf{v}$ ________ be parallel to $\mathbf{x} + \mathbf{y}$.

(d) If E_1, E_2 are elementary matrices, then E_1E_2 ________ also be an elementary matrix.

18. (4 points) Let $T : V \to W$ be a one-to-one linear transformation of vector spaces.

Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set of vectors in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be a linearly dependent set of vectors in V.

ANSWERS

1. (a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

2. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Answers may vary.)

(b) No such matrix exists.

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Answers may vary.)

(d) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Answers may vary.)

3. $\left[\begin{array}{rrr|r} 6 & -4 & 0 & -8 \\ -4 & 8 & -3 & -3 \\ 0 & -3 & 7 & 10 \end{array}\right]$

4. (a) $\frac{2500}{3}$ (b) 27 (c) 0

5. $A^{-1} = \begin{bmatrix} -9 & -29 & 16 \\ 3 & 9 & -5 \\ -1 & -4 & 2 \end{bmatrix}$

6. $LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 4 \\ 0 & -3 & -6 \\ 0 & 0 & 5 \end{bmatrix}$

7. $x_2 = -\frac{9}{20}$

8. (a) $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (b) $A^{87} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

9. (a) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (Answers may vary.)

(b) $\begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) No

10. (a) $\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}$ (Answers may vary.) (b) Yes (c) Yes (d) Yes (e) Yes

11. $\mathcal{B} = \{t^2 - 4, t^3 - 4t\}$ (Answers may vary.)

12. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$ (b) $(-\frac{47}{14}, \frac{12}{7}, -\frac{13}{14})$

(c) $x - 2y - 3z = 1$ (d) $\frac{5}{\sqrt{14}}$ (e) $k = -6, h = 31$

13. (a) $\mathbf{u} = \frac{1}{\sqrt{26}} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ (b) $\sqrt{26}$ (c) $60\sqrt{26}$

14. $4a - 10b + c = 0$

15. (a) $A = I - CB^{-1}$

(b) If B were to equal C, then A would equal 0, which contradicts the assumption that A is invertible.

16. (a) n (b) Yes (c) $\dim(\text{Col}(A^T)) = 3$, $\dim(\text{Row}(A^T)) = 3$, $\dim(\text{Nul}(A^T)) = 1$

17. (a) must (b) must (c) might (d) might

18. Let $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ be a linearly dependent set of vectors in W. Then there must be real numbers a_1, a_2, a_3 not all zero such that $a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + a_3T(\mathbf{v}_3) = \mathbf{0}_W$. Since T is linear, $T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) = \mathbf{0}_W$. Since T is 1-1, the only pre-image of $\mathbf{0}_W$ is $\mathbf{0}_V$, so $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}_V$. This dependence relation on vectors in V tells us that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly dependent.