1. [8 points] Solve each of the following systems or show that the system has no solution. Give parametric solutions where applicable.

(a) $\begin{cases} 2x_1 + 4x_3 = 8 \\ 5x_1 + 6x_2 + 4x_3 = 8 \\ -4x_1 + 4x_2 - 15x_3 = -33 \end{cases}$ (b) $\begin{cases} 2x_1 - 4x_2 - 4x_3 + 2x_4 = 14 \\ 4x_1 - 8x_2 - 7x_3 + 7x_4 = 23 \\ -x_1 + 2x_2 + 5x_3 + 9x_4 = -23 \end{cases}$

2. [3 points] Find the intersection of the planes given by $x + 2y + 7z = 2$ and $2x + 3y + 8z = 2$. Is the intersection a point, a line, a plane, or are the planes parallel?

3. [4 points] Find the standard equation of the plane that contains the points $(1, 0, 3)$, $(2, 2, 7)$ and $(-3, -2, 5)$. (The standard equation of a plane is in the form $Ax + By + Cz = D$.)

4. A company makes toy dinosaurs out of wood and glue. A velociraptor requires 3 units of wood and 3 units of glue. A pterodactyl requires 7 units of wood and 6 units of glue. A T-Rex requires 150 units of wood and 135 units of glue. Suppose that there are 390 units of wood and 360 units of glue available. We would like to determine how many of each kind of dinosaur can be made so that all of the wood and glue is used up.

(a) [2 points] Name variables and set-up a linear system that represents this situation. Do not solve.

(b) [2 points] Find all realistic solutions, given that the general solution to the problem is: $\begin{cases} x_1 = 60 - 15t \\ x_2 = 30 - 15t \\ x_3 = t \end{cases}$

5. [3 points] The following matrix represents a linear system: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & h & k \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Find all values of h and k, if any, such that the system has
(a) a unique solution. (b)infinitely many solutions. (c) no solutions.

6. [7 points] Let $A = \begin{bmatrix} 4 & -2 \\ -3 & 5 \\ 7 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -4 & 7 \\ -5 & 3 & 2 \\ 0 & 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 & 4 \\ 2 & -5 & 3 \end{bmatrix}$, and $D^{-1} = \begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix}$.
Find the following or state that they are undefined.

(a) $CB - A^T$ (b) $CB^{-1}C$ (c) $2D^{-1} - (D^2)^{-1}$ (d) A matrix X such that $2X + 3I = B$.

7. [4 points] Find the determinant of the matrix: $A = \begin{bmatrix} 3 & 2 & 0 & -4 \\ -2 & 8 & 3 & 7 \\ 0 & -5 & -1 & 4 \\ -1 & 2 & 0 & 1 \end{bmatrix}$

8. [3 points] Given $\begin{vmatrix} 3 & -6 & 1 \\ -5 & -3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -69$, use Cramer's rule to solve for x_2 in the following system:

$$\begin{cases} 3x_1 - 6x_2 + x_3 = 2 \\ -5x_1 - 3x_2 + x_3 = 1 \\ 2x_1 + 3x_2 + x_3 = -1 \end{cases}$$

9. [5 points] Let A, B and C be 3×3 matrices with $\det(A) = 3$, $\det(B) = 0$ and $\det(C) = 5$.
(a) Find $\det(-2AC^T)$. (b) Find $\det(A^3C^{-1})$. (c)Is the matrix ABC invertible? Justify.

(d) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, evaluate $\begin{vmatrix} a & b & c \\ g/2 & h/2 & i/2 \\ d-3a & e-3b & f-3c \end{vmatrix}$.

10. [3 points] Suppose A is an $3 \times n$ matrix.

(a) If the rank of A^T is 2 and the nullity of A is 4, find n.

(b) If the nullity of A is n, what is the rank of A?

(c) If $A\mathbf{x} = \mathbf{0}$ has a unique solution, what are the possible values of n?

11. [8 points] Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 3 \\ 5 & 3 & -6 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}$.

(a) Find $\operatorname{adj}(A)$. (b) Find the matrix product $A \cdot \operatorname{adj}(A)$. (c)Find $\det(A)$.
(d) Find A^{-1} using $\operatorname{adj}(A)$. (e) Solve $AX = B$ using A^{-1}.

12. [5 points] A simple economy consists of two industries: Foo and Bar. To produce \$1 of Foo requires 80¢ of Foo and 20¢ of Bar. To produce \$1 of Bar requires 50¢ of Foo and 40¢ of Bar.

(a) Find the production required to meet an external demand for \$120 of Foo and \$180 of Bar.

(b) Determine whether or not each of the industries is profitable. Justify your answer.

(c) Find the internal consumption of the economy.

13. [4 points] Given points $P(1, 0, 3)$ and $Q(2, 0, 1)$.

(a) Find the vector $\overrightarrow{PQ}$.

(b) Find the magnitude of the vector $\overrightarrow{PQ}$.

(c) Find parametric equations for the line L passing through P and Q.

(d) Is the point $R(2, 0, 3)$ on the line L? Justify.

14. [5 points] Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 4 \\ -1 \\ a \end{bmatrix}$.

(a) For what value(s) of a is $\mathbf{w}$ in the span of $\mathbf{u}$ and $\mathbf{v}$?

(b) For the value(s) you found in part (a), express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) For the value(s) you found in the part (a), express $\mathbf{u}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

(d) For what value(s) of a is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a basis for $\mathbb{R}^3$?

15. [9 points] The matrix $A = \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 5 & 1 & 11 & 0 & 5 \\ 4 & 1 & 9 & 1 & 5 \\ -2 & -2 & -6 & 10 & 8 \end{bmatrix}$, reduces to $\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Also let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$,

and let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ represent the columns of the matrix A.

(a) What is the rank of A?

(b) Determine whether each of the following sets is linearly independent or dependent. Justify.

i. $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ ii $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ iii. $\{\mathbf{a}_3, \mathbf{a}_5\}$

(c) Express $\mathbf{a}_4$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_5$ if possible.

(d) Find a basis for $\text{Nul}(A)$.

(e) Find a vector $\mathbf{b}$ such that $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(f) Give the general solution to $A\mathbf{x} = \mathbf{b}$.

16. [6 points] For each set S below, determine if S is a subspace of $\mathbb{R}^3$. If so, find a basis for S. If not, provide a counter-example to show that one of the closure properties fails.

(a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x - 2y + 7z = 0 \right\}$ (b) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; xy \leq 0 \right\}$

17. [8 points] The Picture Perfect Company produces slides, pictures, and posters at two stores with the following daily output and operational cost:

Store	Slides	Pictures	Posters	Cost
1	100	300	150	\$100
2	200	200	100	\$150

The company must produce at least 6000 slides, 3000 pictures, and 6000 posters in at most 60 days. We want to determine how many days each store should be open in order to minimize operating costs.

(a) Define variables for this situation.

(b) State the objective and identify the objective function.

(c) State all constraints in terms of the variables you have defined.

(d) Solve the problem using the graphical method.

18. [6 points] Solve using the simplex algorithm. Give the feasible solution including the slack variables.

$$\begin{array}{llrcrcrl} \text{Maximize} & z = & 7x_1 & + & 6x_2 & + & 4x_3 & \\ \text{subject to} & & 2x_1 & + & 2x_2 & - & x_3 & \leq 6 \\ & & 2x_1 & + & x_2 & & & \leq 4 \\ & & & & 2x_2 & + & x_3 & \leq 2 \\ & & \multicolumn{6}{l}{x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0} \end{array}$$

19. [5 points] The Perfect Doll Company makes three types of dolls. A Baby Doll requires 1 unit of plastic, 2 sheets of cloth, and 1 package of hair; a Boy Doll requires 2 units of plastic, 2 sheets of cloth, and 2 packages of hair; and a Girl Doll requires 2 units of plastic, 4 sheets of cloth, and 4 packages of hair. The company has 1000 units of plastic, 1200 sheets of cloth, and 2400 packages of hair. It makes a profit of \$7, \$8, and \$4 respectively on Baby, Boy, and Girl Dolls. Assuming all dolls are sold, the company would like to determine how many of each doll they should make to maximize profit.

(a) Name variables and set-up a linear program that represents this situation.

Below you will find the initial and final simplex tables for this program. Use this information to answer the remaining questions.

$$\left[\begin{array}{c|ccc|ccc|c} 1 & -7 & -8 & -4 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 2 & 1 & 0 & 0 & 1000 \\ 0 & 2 & 2 & 4 & 0 & 1 & 0 & 1200 \\ 0 & 1 & 2 & 4 & 0 & 0 & 1 & 2400 \end{array}\right] \rightarrow \left[\begin{array}{c|ccc|ccc|c} 1 & 0 & 0 & 10 & 1 & 3 & 0 & 4600 \\ \hline 0 & 0 & 2 & 0 & 2 & -1 & 0 & 800 \\ 0 & 1 & 0 & 2 & -1 & 1 & 0 & 200 \\ 0 & 0 & 0 & 2 & -1 & 0 & 1 & 1400 \end{array}\right]$$

(b) What is the maximum profit?

(c) How many dolls of each type should be made to maximize profit?

(d) When profit is maximized, how much of each material, if any, is left?

Answers:

1.(a) $(-2,1,3)$ 1.(b) $(4+2t,t,-2,-1)$

2. The intersection is a line: $(-2+5t,2-6t,t)$

3. $2x-3y+z=5$ or any non-zero multiple.

4.(a) Let x_1 =the number of velociraptors, $x_2=$#pterodactyls, and $x_3=$#T-rexes.

$$\begin{cases} 3x_1 + 7x_2 + 150x_3 = 390 \\ 3x_1 + 6x_2 + 135x_3 = 360 \end{cases}$$

4.(b) $(60,30,0),(45,15,1),(30,0,2)$.

5.(a) No values of k. This case is impossible.

5. (b) $k=0$. 5.(c) $k\neq 0$. (Note: h is irrelevant.)

6.(a) $\begin{bmatrix}12 & -3 & 30\\ 31 & -19 & 19\end{bmatrix}$ 6.(b) Undefined.

6.(c) $\begin{bmatrix}-62 & -16\\ -56 & -14\end{bmatrix}$ 6.(d) $X=\begin{bmatrix}-1/2 & -2 & 7/2\\ -5/2 & 0 & 1\\ 0 & 3/2 & 1\end{bmatrix}$

7. 125 8. $x_2=-1/3$

9.(a) -120 9.(b) $27/5$

9.(c) No. $\det(ABC)=0$. 9.(d) $-3/2$.

10.(a) 6 10.(b) 0 10.(c) $n=1,2$, or 3.

11.(a) $\text{adj}(A)=\begin{bmatrix}-21 & 6 & 3\\ 33 & -12 & -6\\ -1 & -1 & 1\end{bmatrix}$

11.(b) $A\cdot\text{adj}(A)=\begin{bmatrix}-9 & 0 & 0\\ 0 & -9 & 0\\ 0 & 0 & -9\end{bmatrix}$ 11.(c) -9

11.(d) $A^{-1}=\begin{bmatrix}7/3 & -2/3 & -1/3\\ -11/3 & 4/3 & 2/3\\ 1/9 & 1/9 & -1/9\end{bmatrix}$ 11.(e) $X=\begin{bmatrix}-2\\ 4\\ 1\end{bmatrix}$

12.(a) \$8100 of Foo and \$3000 of Bar.

12.(b) Foo is not profitable as it costs \$1 to produce \$1 of Foo. Bar is profitable as it costs \$.90 to produce \$1 of Bar.

12.(c) \$7980 of Foo and \$2820 of Bar.

13.(a) $\overrightarrow{PQ}=(1,0,-2)$ 13.(b) $\sqrt{5}$

13.(c) $\begin{cases} x = 1+t\\ y = 0\\ z = 3-2t\end{cases}$

13.(d) No. There is no value of t that will give the point R.

14.(a) $a=-5$ 14.(b) $\mathbf{w}=-2\mathbf{u}+3\mathbf{v}$

14.(c) $\mathbf{u}=\frac{3}{2}\mathbf{v}-\frac{1}{2}\mathbf{w}$ 14.(d) $a\neq -5$

15.(a) 3 15.(b) i. LD. $\mathbf{a}_3=2\mathbf{a}_1+\mathbf{a}_2$

15.(b) ii. LI. Rank$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_5]=3$

15.(b) iii. LI. The two vectors are not proportional.

15.(c) $\mathbf{a}_4=-1\mathbf{a}_1+0\mathbf{a}_2+1\mathbf{a}_5$

15.(d) $\left\{\begin{bmatrix}-2\\ -1\\ 1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 0\\ -1\\ 1\end{bmatrix}\right\}$ 15.(e) $\mathbf{b}=\begin{bmatrix}9\\ 12\\ 13\\ 22\end{bmatrix}$

15.(f) $\mathbf{x}=\begin{bmatrix}1\\ 2\\ 0\\ 2\\ 1\end{bmatrix}+s\begin{bmatrix}-2\\ -1\\ 1\\ 0\\ 0\end{bmatrix}+t\begin{bmatrix}-1\\ 0\\ 0\\ -1\\ 1\end{bmatrix}$

16.(a) Yes, it is a subspace. One basis for S is $\left\{\begin{bmatrix}2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-7\\ 0\\ 1\end{bmatrix}\right\}$.

16.(b). No, it's not a subspace because S is not closed under vector addition. Counter-example: $\mathbf{u}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$. $\mathbf{u}\in S$ and $\mathbf{v}\in S$, but $\mathbf{u}+\mathbf{v}\notin S$.

17.(a) Let $x=$ # days store 1 is open.
Let $y=$ # days store 2 is open.

17.(b) Minimize $z=100x+150y$.

17.(c) subject to: $\begin{cases} 100x + 200y \geq 6000\\ 300x + 200y \geq 3000\\ 150x + 100y \geq 6000\\ 0\leq x\leq 60\\ 0\leq y\leq 60\end{cases}$

17.(d) Minimum cost is \$5250 when store 1 is open 30 days, and store 2 is open 15 days.

18. Max $z=22$ at $(2,0,2,4,0,0)$.

19.(a) Let $x_1=$#Baby Dolls, $x_2=$# Boy Dolls, and $x_3=$#Girl Dolls.

$$\begin{array}{llcccccl} \text{Maximize} & P = 7x_1 & + & 8x_2 & + & 4x_3 & & \\ \text{subject to} & x_1 & + & 2x_2 & + & 2x_3 & \leq & 1000\\ & 2x_1 & + & 2x_2 & + & 4x_3 & \leq & 1200\\ & x_1 & + & 2x_2 & + & 4x_3 & \leq & 2400\\ & \multicolumn{7}{l}{x_1\geq 0,\quad x_2\geq 0,\quad x_3\geq 0} \end{array}$$

19.(b) Max profit is \$4600.

19.(c) Max profit occurs when 200 Baby Dolls, 400 Boy Dolls, and 0 Girl Dolls are made.

19.(d) When profit is maximized there are 0 units of plastic, 0 sheets of cloth, and 1400 packages of hair left over.