1. (35 points) Evaluate the following integrals.

(a) $\displaystyle\int_{\sqrt{2}}^{2} \frac{3}{x\sqrt{x^2-1}}\,dx$

(b) $\displaystyle\int \frac{xe^x}{\sqrt{xe^x-e^x}}\,dx$

(c) $\displaystyle\int \frac{\ln x}{x^3}\,dx$

(d) $\displaystyle\int \sec x \tan^5 x\,dx$

(e) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

(f) $\displaystyle\int \frac{x-6}{x^3(x-2)}\,dx$

(g) $\displaystyle\int_0^{\pi/2} \frac{\cos^3 x}{1+\sin^2 x}\,dx$

2. (6 points) Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} x\left(\arctan(3x) - \pi/2\right)$

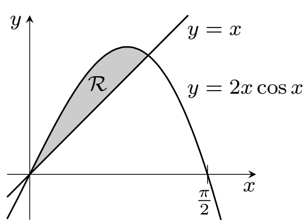
(b) $\displaystyle\lim_{x\to 0^+} (\sec x + \tan x)^{1/x}$

3. (8 points) Evaluate each improper integral or show it diverges.

(a) $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_3^{\infty} \frac{1}{\sqrt[5]{x-2}}\,dx$

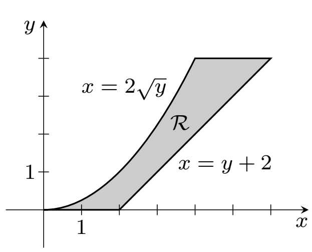
4. (5 points) The figure below shows the graphs of the functions $y = 2x\cos x$ and $y = x$. Find the area of the shaded region $\mathcal{R}$.



5. (4 points) Let $\mathcal{R}$ be the region bounded by $x = 2\sqrt{y}$, $x = y + 2$, $y = 0$, and $y = 4$ (see figure). Set up, but **do not evaluate**, the integral for the volume of the solid obtained by rotating $\mathcal{R}$ around:

(a) the x-axis

(b) the vertical line $x = -1$



6. (5 points) Find the length of the curve $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, $1 \leqslant x \leqslant 2$.

7. (5 points) Express y as a function of x if $\dfrac{dy}{dx} = \dfrac{x}{x^2y+y}$ and $y = -4$ if $x = 0$.

8. (3 points) Let $f(x) = \sin(\frac{1}{2}x)$. Let $\{a_1, a_2, a_3, \dots\}$ be the sequence defined by $a_n = f^{(n)}(0)$, where $f^{(n)}$ is the n^{th} derivative of f.

(a) Find the first five terms of $\{a_n\}$.

(b) Does the sequence converge or diverge? If it converges, find the limit. Justify your answer.

9. (3 points) Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be the series whose n^{th} partial sum is $s_n = \dfrac{5n}{2n+1}$.

(a) Evaluate $\displaystyle\sum_{n=1}^{\infty} a_n$.

(b) Find a_3.

10. (9 points) Determine whether the series converges or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{3n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + 7^n}{2^n + 9^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt{n+1}}$

11. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

(a) (3 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{(2n)!}$

(b) (4 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

12. (4 points) Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(2x-3)^n}{3^n\sqrt{2n+3}}$.

13. (4 points) Find the Maclaurin series of $f(x) = \dfrac{1}{\sqrt{2x+1}}$.

14. (2 points) (a) Given that $\sum a_n$ is a convergent series of *positive* terms, prove that $\sum (a_n)^2$ is also convergent.

(b) Give an example of a series $\sum a_n$ such that $\sum a_n$ is convergent but $\sum (a_n)^2$ is divergent.

Answers

1. (a) $3\operatorname{arcsec} x \Big]_{\sqrt{2}}^{2} = \frac{1}{4}\pi$ (b) $2\sqrt{xe^x - e^x} + C$

(c) $-\dfrac{2\ln x + 1}{4x^2} + C$

(d) $\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$

(e) $8\arcsin(\frac{1}{4}x) - \frac{1}{2}x\sqrt{16 - x^2} + C$

(f) $\frac{1}{2}\ln|x| - \dfrac{1}{x} - \dfrac{3}{2x^2} - \frac{1}{2}\ln|x - 2| + C$

(g) Letting $u = \sin x$, the integral equals

$$2\arctan u - u \Big]_0^1 = \tfrac{1}{2}\pi - 1$$

2. (a) $-\frac{1}{3}$ (b) e

3. (a) Converges to $2(e-1)$

(b) Diverges (to ∞)

4. $\displaystyle\int_0^{\pi/3} (2x\cos x - x)\,dx = \tfrac{1}{3}\pi\sqrt{3} - 1 - \tfrac{1}{18}\pi^2$

5. (a) $\displaystyle\int_0^4 2\pi y(y + 2 - 2\sqrt{y})\,dy$

(b) $\displaystyle\int_0^4 \pi\big[(y+3)^2 - (2\sqrt{y} + 1)^2\big]\,dy$

6. $\frac{13}{12}$

7. $y = -\sqrt{\ln(x^2+1) + 16}$

8. (a) $\{a_n\} = \{\frac{1}{2}, 0, -\frac{1}{8}, 0, \frac{1}{32}, \ldots\}$

(b) Since $a_n = 0$ if n is even and $a_n = (-1)^{(n-1)/2}(\frac{1}{2})^n$ if n is odd, we clearly have

$$0 \leqslant |a_n| \leqslant (\tfrac{1}{2})^n \quad \text{for all } n$$

Now apply the Squeeze Theorem:

$$\lim_{n\to\infty} (\tfrac{1}{2})^n = 0 \implies \lim_{n\to\infty} |a_n| = 0$$

and conclude that $\lim_{n\to\infty} a_n = 0$ as well.

9. (a) $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \tfrac{5}{2}$

(b) $a_3 = s_3 - s_2 = \frac{1}{7}$

10. Let a_n be the nth term of the series in question.

(a) Diverges by the test for divergence:

$$a_n = \left[\left(1 + \frac{1}{n}\right)^n\right]^3 \longrightarrow e^3 \neq 0 \quad \text{as } n \to \infty$$

(Or simply note that since $a_n > 1$ for all n, we clearly have $\lim_{n\to\infty} a_n \neq 0$.)

(b) Converges by the direct comparison test:

$$a_n = \frac{5^n}{2^n + 9^n} + \frac{7^n}{2^n + 9^n} < \frac{5^n}{9^n} + \frac{7^n}{9^n} = (\tfrac{5}{9})^n + (\tfrac{7}{9})^n = b_n$$

and $\sum b_n$ converges because it is the sum of two convergent geometric series ($|\frac{5}{9}| < 1$, $|\frac{7}{9}| < 1$). Alternatively, we can use the limit comparison test. Let $b_n = (\frac{7}{9})^n$. Then $\sum b_n$ converges and

$$\frac{a_n}{b_n} = \frac{(\frac{5}{7})^n + 1}{(\frac{2}{9})^n + 1} \longrightarrow 1 \neq 0, \infty \quad \text{as } n \to \infty$$

(c) Converges by the direct comparison test. Since $-1 \leqslant \cos n \leqslant 1$, $\cos^2 n \leqslant 1$. Therefore

$$a_n \leqslant \frac{1}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}} = b_n$$

and $\sum b_n$ is a convergent p-series ($p = \frac{3}{2} > 1$).

11. Let a_n be the nth term of the series in question.

(a) Converges absolutely by the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{(2n+2)(2n+1)} \longrightarrow 0 < 1$$

(b) Converges conditionally. $\sum |a_n| = \sum \sin(1/n)$ diverges by limit comparison with $\sum 1/n$:

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0, \infty$$

On the other hand, $\sin(1/n) \to 0$ as $n \to \infty$ and is decreasing, so $\sum a_n$ converges by the alternating series test.

12. $R = \frac{3}{2}$, $[0, 3)$

13. $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}\, x^n$

14. (a) Since $\sum a_n$ is convergent, $a_n \to 0$ as $n \to \infty$, and so $a_n \leqslant 1$ for all sufficiently large n. Multiplying both sides of this inequality by $a_n > 0$ shows that

$$(a_n)^2 \leqslant a_n \quad \text{for all sufficiently large } n,$$

so $\sum (a_n)^2$ converges by direct comparison with $\sum a_n$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges (alternating series test) but

$$\sum_{n=1}^{\infty} \Big[\frac{(-1)^n}{\sqrt{n}}\Big]^2 = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.