1. (8 points) Solve each of the following systems or show that it is inconsistent:

(a) $\begin{cases} x_1 + 2x_2 - 2x_3 = 3 \\ 2x_1 - x_2 + x_3 = 1 \\ -x_1 + 8x_2 - 6x_3 = 5 \end{cases}$

(b) $\begin{cases} 2x_1 + 16x_2 - 5x_3 = 5 \\ 4x_2 + x_3 = 3 \\ x_1 + 6x_2 - 3x_3 = 2 \end{cases}$

2. (5 points) Find the value of the constant k such that the system
$\left\{\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & - & x_4 & = & 2 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 3 \\ 3x_1 & + & 4x_2 & - & 4x_3 & + & kx_4 & = & 8 \end{array}\right\}$ is <u>inconsistent</u>.

3. (5 points) Let P be a plane that passes through the points $A(2,4,1)$, $B(3,-10,0)$ and $C(5,2,2)$. Write an equation for P in standard form.

4. (6 points) Abby has a fruit stand that sells pineapples, coconuts, and watermelons. The pineapples sell for $2 each, the coconuts for $4 each, and the watermelons for $5 each. She sold 24 fruits yesterday, and her revenue was $70.

(a) Define the necessary variables and set up the system of equations in order to solve the problem.

(b) Find a general (parametric) solution to the system.

(c) Find all realistic solutions.

5. (5 points) Suppose that A is a 4×4 matrix with $\det(A)=4$, and B is a 4×4 matrix with $\det(B)=-3$ and C is a 4×4 matrix with $\det(C)=7$. Find:

(a) $\det(A^T)$

(b) $\det(B^4)$

(c) $\det(2C)$

(d) $\det(ABC)$

(e) $\det(AB^{-1})$

6. (3 points) Use Cramer's rule to solve the linear system: $\begin{cases} 3x - 4y = 11 \\ -x - 2y = -2 \end{cases}$

7. (7 points) Let $A = \begin{bmatrix} 1 & 7 & -3 \\ -3 & 4 & -1 \\ 2 & 5 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

(a) Find $\det(A)$

(b) Find $\mathrm{adj}(A)$

(c) Find A^{-1} using (a) and (b).

 Question 7 continues on the next page.

(d) Using your answer from (c), solve the equation $AX = B$.

8. (4 points) Consider the line $L_1 : \{x = 2 + 2t, y = 3 - 6t, z = 2 - 6t, t \in \mathbb{R}\}$, and the points $P(4, -3, 2)$, $Q(3, 0, 5)$ and $R(2, 7, -4)$

(a) Find the magnitude of $\overrightarrow{PQ}$

(b) Is the point P on L_1?

(c) Find a vector equation of the line L_2 that is parallel to the vector $\overrightarrow{PQ}$ and that passes through R

9. (5 points) Find the determinant of $\begin{bmatrix} 4 & 2 & 7 & -3 \\ -1 & 0 & 5 & 8 \\ 5 & 0 & 4 & -1 \\ 3 & -4 & 2 & 0 \end{bmatrix}$.

10. (4 points) For which value(s) of k would the vector $\overrightarrow{v} = (6, k, -3)$ be a linear combination of the vectors $\overrightarrow{a_1} = (2, -3, 5)$ and $\overrightarrow{a_2} = (4, -2, 1)$?

11. (3 points) Show that the points $P(2, 3, 5), Q(1, 6, 9)$ and $R(-3, 18, 25)$ all lie on the same line.

12. (6 points) An economy is developed around the activities of two industries: gas and transportation. For each dollar of gas producted, 20 worth of gas and 10 worth of transportation is consumed. For each dollar of transportation produced, 30 worth of gas and 40 worth of transportation is consumed.

(a) If there is an external demand for $\$2,000,000$ in gas and $\$650,000$ in transportation, how much gas and transportation should be produced?

(b) What is the internal consumption?

(c) Which if any industries is profitable? Justify your answer.

(d) Which, if any, of the industries are profitable? Justify your answer.

(e) Is the economy productive? Justify your answer.

13. (6 points) Let $S = \{(x, y, z) \in \mathbb{R}^3 | x - 7z = 0\}$.

(a) Is $\mathbf{0}$ in S? Justify your answer.

(b) Give two non-zero vectors in S.

(c) Is S closed under vector addition? Justify your answer.

(d) Is S closed under scalar multiplication? Justify your answer.

(e) Is S a subspace of $\mathbb{R}^3$?

14. (8 points) Let $A = \begin{bmatrix} -1 & 3 & 0 & -1 & -6 \\ 4 & -12 & 0 & 2 & 20 \\ -3 & 9 & 0 & 0 & -12 \end{bmatrix}$ where $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}$, and $\mathbf{a_5}$ are the columns of A.

A reduces to $R = \begin{bmatrix} 1 & -3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

 Question 14 continues on the next page.

(a) Find a basis for $\mathrm{Col}(A)$

(b) For each of the following sets, state whether it is linearly dependent or linearly independent. Briefly justify your answer.
(*i*) $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_4}\}$ (*ii*) $\{\mathbf{a_3}\}$ (*iii*) $\{\mathbf{a_2}, \mathbf{a_5}\}$

(c) Find a basis for $\mathrm{Nul}(A)$.

(d) Give the rank of A and the nullity of A.

15. (3 points) Suppose A is a 7×5 matrix.

(a) Is it possible for $A\mathbf{x} = \mathbf{0}$ to have a unique solution? Explain.

(b) Is it possible for $A^T\mathbf{y} = \mathbf{0}$ to have a unique solution? Explain.

(c) If $\mathrm{rank}(A) = 0$, what is the nullspace $\mathrm{Nul}(A)$?

16. (8 points) A blade company manufactures scissors and knives made from stainless steel. Production of a single pair of scissors requires 0.4kg of steel and requires 0.3 man-hours of labour. Production of a single knife requires 0.5kg of steel and requires 0.2 man-hours of labour. The number of employees working for the company allows them to use up to 850 man-hours per week and the companys producer can deliver up to 1,250kg of stainless steel per week. If a pair of scissors will generate \$9 in profit for the company, whereas a knife will generate \$11 in profit, what is the maximum weekly profit that the company can achieve?

(a) Define all necessary variables.

(b) State the objective function in terms of these variables.

(c) State all the constraints in terms of these variables.

(d) Solve the problem using the graphical method.

17. (7 points) Consider the function $z = -17x_1 - 67x_2 + 6x_3$

$$\text{subject to} \quad \begin{array}{rcrcrcr} -27x_1 & - & 8x_2 & + & 8x_3 & \leqslant & 17 \\ -7x_1 & - & 19x_2 & + & 2x_3 & \leqslant & 8 \\ -15x_1 & - & 10x_2 & - & 5x_3 & \leqslant & 10 \\ x_1 \geqslant 0, & & x_2 \geqslant 0, & & x_3 \geqslant 0 & & \end{array}$$

(a) Use the simplex method to demonstrate that the objective function z has no maximum.

(b) Find a feasible solution for which $z \geqslant 10,000$ to demonstrate that the value of z can be made arbitrarily large.

18. (7 points) Use the simplex method to minimize z below. Your solution should include the minimum value and its corresponding feasible solution.

$$\text{Minimize} \quad z = 15x_1 + 48x_2 - 8x_3 \quad \text{subject to} \left\{ \begin{array}{rcrcrcr} -11x_1 & - & 33x_2 & + & 6x_3 & \leqslant & 7 \\ -4x_1 & - & 12x_2 & + & 2x_3 & \leqslant & 2 \\ -36x_1 & - & 101x_2 & + & 20x_3 & \leqslant & 26 \\ x_1 \geqslant 0, & & x_2 \geqslant 0, & & x_3 \geqslant 0 & & \end{array} \right. .$$

 Question 18 continues on the next page.

Answers

1. a) $(1,0,-1)$ b) Inconsistent 2. $k=-7$ 3. $4x+y-10z=2$
4. $\left(13+\frac{t}{2}, 11-\frac{3t}{2}, t\right)$, Realistic solutions: $(13,11,0),(14,8,2),(15,5,4),(16,2,6)$
5. a) 4 b) 81 c) 112 d) -84 e) $-\frac{4}{3}$ 6. $x=3=, y=-\frac{1}{2}$
7. a) $|A|=210$ b) $adj(A)=\begin{bmatrix} 29 & -57 & 5 \\ 16 & 12 & 10 \\ -23 & 9 & 25 \end{bmatrix}$ c) $A^{-1}=\frac{1}{210}\begin{bmatrix} 29 & -57 & 5 \\ 16 & 12 & 10 \\ -23 & 9 & 25 \end{bmatrix}$
d) $x_1=-\frac{121}{105}, x_2=\frac{31}{105}, x_3=\frac{67}{105}$
8. a) $\sqrt{19}$ b) No. c) $L_2:\begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}+t\begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$ 9. -782 10. $k=-1$
11. It is enough to show that R is on the line $\overline{PQ}$.
12. a) $X=\begin{bmatrix} 3100000 \\ 1600000 \end{bmatrix}$ b) $CX=\begin{bmatrix} 1100000 \\ 950000 \end{bmatrix}$ c) Both gas and transportation are profitable.
   d) Yes, since both industries are profitable, or since $(I-C)^{-1}\geqslant 0$.
13. a) Yes, $0-7(0)=0$.
   b) Any vectors where the x value is 7 times the z-value, e.g. $(7,2,1),(14,0,2)$
   c) Let $\mathbf{u}=(7a,b,a)$ and $v=(7d,f,d)$ be vectors in S. Show that for $\mathbf{w}=\mathbf{u}+\mathbf{v}$, we have $x-7z=0$.
   d) Show that for the vector $c\mathbf{u}$, we have $x-7z=0$ (since $x=7ac$ and $z=ac$).
   e) Yes, it is a subspace.
14. a) $B=\{\mathbf{a_1},\mathbf{a_4}\}$
   b) (i) Linearly Dependent (ii) Linearly Dependent (iii) Linearly Independent
   c) $B=\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} -4 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$
   d) rank $=2$, nullity $=3$
15. a) Yes, if A has 5 pivots, then nullity$(A)=0$. b) No, nullity$(A)\geqslant 2$ c) Nul$(A)=\mathbb{R}^5$
16. a) Let $x=$ number of pairs of scissors, $y=$ number of knives
   b) Maximize $z=9x+11y$ c) subject to $\begin{cases} 0.4x+0.5y\leqslant 1250 \\ 0.3x+0.2y\leqslant 850 \\ x\geqslant 0 \text{ and } y\geqslant 0 \end{cases}$
   d) The maximum profit is \$28000 at $(2500,500)$
17. $z=10000$ at $(3073,0,10373.5,0,772,97972.5)$
18. $z=-9$ at $(1,0,3,0,0,2)$