(4) 1. Use linear algebra to balance the chemical equation:

$$NH_3 + Br_2 \longrightarrow NH_4Br + N_2$$

2. In this problem you are given the matrix $A = [\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{a}_4\ \ \mathbf{a}_5\ \ \mathbf{a}_6]$, with the fourth column unspecified, and **its reduced echelon form** R:

$$A = \begin{bmatrix} 1 & -7 & 4 & a & 5 & 2 \\ -1 & 7 & 2 & b & 2 & -1 \\ 2 & -14 & 3 & c & 3 & 2 \\ 3 & -21 & -2 & d & 1 & 6 \end{bmatrix}; \quad R = \begin{bmatrix} 1 & -7 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(3) (a) Solve the equation $A\mathbf{x} = \mathbf{0}$.

(2) (b) What is $\mathbf{a}_4$? (That is, find a, b, c and d.)

(2) (c) Suppose the vector $\mathbf{b}$ is defined by $\mathbf{b} = \mathbf{a}_2 - 2\mathbf{a}_4 + 3\mathbf{a}_6$. Find the general solution of $A\mathbf{x} = \mathbf{b}$. (Note: No row reduction is necessary.)

(1) (d) What is the dimension of $\operatorname{Nul} A^T$?

3. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 3 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = A\mathbf{x}$. Let $\mathcal{L}$ be the line defined by $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

(1) (a) What are the domain and codomain of T?

(2) (b) What is the range of T? Be as specific as possible!

(2) (c) Find $T(\mathcal{L})$.

(2) (d) More generally, let $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, where $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p} \in \mathbb{R}^n$, be a line in $\mathbb{R}^n$. Let $S : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that S maps this line onto another line or onto a single point (a degenerate line) in $\mathbb{R}^m$.

4. Let $A = \begin{bmatrix} 2 & 1 \\ c & d \end{bmatrix}$.

(2) (a) Find c and d so that $A^2 = 0$.

(2) (b) Find c and d so that $A^2 = I$.

(2) (c) Find all c and d so that A^2 is symmetric.

5. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 15 & 0 & 1 \end{bmatrix}$.

(4) (a) Find the inverse of A.

(4) (b) Write A as a product of elementary matrices.

(6) 6. Let A and B be invertible $n \times n$ matrices. Let C and D be non-invertible $n \times n$ matrices. Fill in the blanks. The missing word is **must**, **might** or **cannot**.

(a) $A + C$ ________ be invertible.

(b) $A^T A$ ________ be invertible.

(c) AC and BC __________ have the same determinant.

(d) $\operatorname{Col} C$ __________ be equal to $\operatorname{Col} D$.

(e) $\operatorname{Nul} A$ __________ be equal to $\operatorname{Nul} B$.

(f) The columns of D __________ be linearly independent.

(4) 7. Let $A = \begin{bmatrix} 3 & -1 & 3 \\ 15 & -3 & 13 \\ 12 & 2 & 10 \end{bmatrix}$. Find an LU factorization of A.

8. Let A be a 5×5 matrix with $\det A = 2$ and let I be the 5×5 identity matrix. Furthermore, assume $A = LU$ where L is unit lower triangular and U is upper triangular. Calculate:

(1) (a) $\det U$

(2) (b) $\det(3A^{-1}A^T)$

(2) (c) $\det(L + I)$

(4) 9. Verify that Cramer's Rule applies to the following system and then use it to solve for x_2 only.

$$\begin{aligned} 3x_1 \qquad\quad - x_3 \qquad\quad &= 2 \\ 2x_2 \qquad\quad + 5x_4 &= 0 \\ -4x_1 \qquad\quad + 2x_3 \qquad\quad &= -1 \\ -5x_2 \qquad\quad - 5x_4 &= 4 \end{aligned}$$

(7) 10. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2 and $M_{2\times 2}$ be the vector space of all 2×2 matrices. Determine whether the following sets are subspaces. If a set is a subspace, find a basis for it. If a set is not a subspace, explain why not.

(a) $\mathcal{S}_1 = \{\mathbf{p}(x) \in \mathbb{P}_2 \ : \ \mathbf{p}(0) \geq 0\}$

(b) $\mathcal{S}_2 = \left\{ A \in M_{2\times 2} \ : \ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A = A \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

11. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \end{bmatrix}.$$

(2) (a) If $\mathbf{p}(x) = x^2 - 1$, find $T(\mathbf{p})$.

(4) (b) Find a basis for the kernel of T.

12. Suppose A is a 5×5 matrix of rank 3. Let $B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

(1) (a) What is the rank of B?

(1) (b) What is the dimension of $\operatorname{Nul} B$?

13. Let $\mathcal{L}_1$ be the line in $\mathbb{R}^3$ defined by $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and let $\mathcal{L}_2$ be the line defined by

$\mathbf{x} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(4) (a) Find the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(2) (b) Find the cosine of the angle between $\mathcal{L}_1$ and $\mathcal{L}_2$.

(4) (c) Find an equation of the form $ax + by + cz = d$ for the plane containing $\mathcal{L}_1$ and $\mathcal{L}_2$.

14. Let $\mathcal{P}$ be the plane $x - 2y + 2z = 8$.

(2) (a) Find the distance from the origin to $\mathcal{P}$.

(2) (b) Find an equation for the line through the origin perpendicular to $\mathcal{P}$.

(2) (c) Find the point on $\mathcal{P}$ closest to the origin.

(3) 15. Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$ such that $\text{Proj}_{\mathbf{w}}(\mathbf{u}+\mathbf{v}) = \text{Proj}_{\mathbf{w}}\mathbf{u}$. Show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(3) 16. (a) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Find two different vectors $\mathbf{w} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ such that the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is 10.

(2) (b) Suppose the volume of the parallelepiped in $\mathbb{R}^3$ formed by three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is 6. What is the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{a}+\mathbf{b}$, and $\mathbf{a}+\mathbf{b}+\mathbf{c}$?

(2) 17. (a) Is the following statement True of False? Justify your answer!
If $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are both linearly **dependent** sets in $\mathbb{R}^n$, then either $\{\mathbf{a}, \mathbf{u}\}$ or $\{\mathbf{a}, \mathbf{v}\}$ must be linearly **dependent**.

(2) (b) Suppose that $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are both linearly **independent** sets in $\mathbb{R}^n$. Show that either $\{\mathbf{a}, \mathbf{u}\}$ or $\{\mathbf{a}, \mathbf{v}\}$ must be linearly **independent**.

18. Let $A = \begin{bmatrix} I & \mathbf{u} \\ \mathbf{u}^T & 0 \end{bmatrix}$, where I is the $n \times n$ identity matrix and $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$. (Note: You will need to use the fact that $\mathbf{u}$ is a unit vector in simplifying the following.)

(2) (a) Find A^2.

(3) (b) Find A^{-1}.

Answers

1. $8\text{NH}_3 + 3\,\text{Br}_2 \longrightarrow 6\,\text{NH}_4\text{Br} + \text{N}_2$

2. (a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ where $r, s, t \in \mathbb{R}$

(b) $\mathbf{a}_4 = -3\mathbf{a}_1 + 2\mathbf{a}_3 = \begin{bmatrix} 5 \\ 7 \\ 0 \\ -13 \end{bmatrix}$

(c) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \\ 3 \end{bmatrix} + r\begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{where } r, s, t \in \mathbb{R}$$

(d) $\dim(\operatorname{Nul} A^T) = 1$

3. (a) domain= $\mathbb{R}^3$ and codomain= $\mathbb{R}^2$

   (b) range of T= $\operatorname{Col} A = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

   (c) $T(\mathcal{L}) : T(\mathbf{x}) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

   (d) $S(\mathbf{x}) = S(\mathbf{p} + t\mathbf{v}) = S(\mathbf{p}) + tS(\mathbf{v})$; if $S(\mathbf{v}) = \mathbf{0}$ then $S(\mathbf{x}) = S(\mathbf{p})$ is a point in $\mathbb{R}^m$; if $S(\mathbf{v}) \neq \mathbf{0}$ then $S(\mathbf{x})$ is a line in $\mathbb{R}^m$.

4. (a) $c = -4$ and $d = -2$

   (b) $c = -3$ and $d = -2$

   (c) Either $d = -2$ and $c \in \mathbb{R}$, or $c = 1$ and $d \in \mathbb{R}$

5. (a) $A^{-1} = \begin{bmatrix} 0 & 1/3 & 0 \\ 1 & 0 & 0 \\ 0 & -5 & 1 \end{bmatrix}$.

   (b) Many answers possible, e.g., $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6. (a) might

   (b) must

   (c) must

   (d) might

   (e) must

   (f) cannot

7. $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

8. (a) 2

   (b) $3^5 = 243$

   (c) $2^5 = 32$

9. $\det A = 30 \neq 0$ so Cramer's Rule applies and $x_2 = -\frac{4}{3}$.

10. (a) $\mathcal{S}_1$ is not closed under scalar multiplication; e.g. $p_1(x) = 1 + x^2 \in \mathcal{S}_1$ but $-p_1(x) = -1 - x^2 \notin \mathcal{S}_1$; so it is not a subspace of $\mathbb{P}_2$.

(b) $\mathcal{S}_2 = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ so it is a subspace of $M_{2\times 2}$;
a basis for $\mathcal{S}_2$ is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$

11. (a) $T(x^2-1) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

    (b) A basis for the kernel of T is $\mathcal{B} = \{2 - 3x + x^2\}$.

12. (a) $\operatorname{rank} B = 3$

    (b) $\dim(\operatorname{Nul} B) = 7$

13. (a) $(3, 4, 5)$

    (b) $\cos\theta = \frac{4}{\sqrt{66}} = \frac{2\sqrt{66}}{33}$

    (c) $4x + 5y - 3z = 17$

14. (a) distance$=D = \frac{8}{3}$

    (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

    (c) $(\frac{8}{9}, \frac{-16}{9}, \frac{16}{9})$

15. $\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} = \left(\frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} \longrightarrow \left(\frac{\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}} - \frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} = \mathbf{0}$

    Since $\mathbf{w} \neq 0$, this implies that $\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}} = 0$ which implies $\mathbf{v}\cdot\mathbf{w} = 0$

16. (a) $k = \pm\frac{10}{9} \longrightarrow \mathbf{w} = \begin{bmatrix} \pm\frac{10}{9} \\ 0 \\ 0 \end{bmatrix}$

    (b) $|\mathbf{a}\ \ \mathbf{a}+\mathbf{b}\ \ \mathbf{a}+\mathbf{b}+\mathbf{c}| = |\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}|$ so the required volume is also 6

17. (a) False, e.g., let $\{\mathbf{a},\mathbf{b}\} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$ and $\{\mathbf{u},\mathbf{v}\} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}\right\}$ then both $\{\mathbf{a},\mathbf{u}\} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $\{\mathbf{a},\mathbf{v}\} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}\right\}$ are linearly independent.

    (b) Note that $\mathbf{a}, \mathbf{b}, \mathbf{u}$ and $\mathbf{v}$ are all nonzero. If $\{\mathbf{a},\mathbf{u}\}$ is linearly independent then there is nothing to prove. If $\{\mathbf{a},\mathbf{u}\}$ is linearly dependent then $\mathbf{a} = k\mathbf{u}$ for some $k \neq 0$. Therefore, $\{\mathbf{a},\mathbf{v}\} = \{k\mathbf{u},\mathbf{v}\}$ which is linearly independent since $\{\mathbf{u},\mathbf{v}\}$ is linearly independent.

18. (a) $A^2 = \begin{bmatrix} I + \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & 1 \end{bmatrix}$

    (b) $A^{-1} = \begin{bmatrix} I - \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & -1 \end{bmatrix}$