1. Given the following homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 0 & 2 & -1 & 0 \\ 1 & 1 & -5 & 5 & 1 \\ 2 & 2 & -10 & 10 & 3 \\ 2 & 1 & -7 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[4] (a) Write the solution to the system in parametric vector form.

[1] (b) Write the zero vector in $\mathbb{R}^4$ as a nontrivial linear combination of the columns of A.

[4] 2. Use techniques of linear algebra to find a polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ such that $p(2) = 0$, $p(-2) = 32$ and $p'(1) = -7$.

[3] 3. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ k \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} k \\ 0 \\ 2k+3 \end{bmatrix}$ and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. For what value(s) of k is:

(a) Span(S) all of $\mathbb{R}^3$?

(b) Span(S) a plane in $\mathbb{R}^3$?

(c) Span(S) a line in $\mathbb{R}^3$?

4. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x + 2y \\ 2x - 3y \end{bmatrix}$.

[1] (a) Find the standard matrix for T_1.

[3] (b) If $\mathcal{L}$ is the line $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ k \end{bmatrix}$, then for what value(s) of k, will $T_1(\mathcal{L})$ be a horizontal line in $\mathbb{R}^2$?

[3] (c) Now suppose that the composition $T_1 \circ T_2$ is also a linear transformation whose standard matrix is $\begin{bmatrix} 1 & -2 & -3 \\ -3 & 5 & 7 \end{bmatrix}$.

i. Identify the domain and codomain of T_2.

ii. Find the standard matrix for T_2.

[3] 5. Suppose that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, and that $\mathbf{x} = 2\mathbf{u} + 3\mathbf{w}$ and $\mathbf{y} = \mathbf{v} + 2\mathbf{w}$. Prove that the set $\{\mathbf{u}, \mathbf{x}, \mathbf{y}\}$ is linearly independent.

[3] 6. Let $A = \begin{bmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}$. Find an LU factorization of A, where L is unit lower triangular and U is upper triangular.

[4] 7. Let $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}$ and $B = \begin{bmatrix} a+2b+4c & d+2e+4f & g+2h+4k \\ 3a+4b+7c & 3d+4e+7f & 3g+4h+7k \\ 5a+7b+8c & 5d+7e+8f & 5g+7h+8k \end{bmatrix}$

(a) Find a matrix C such that $B = CA$.

(b) Find the value of λ such that $\det B = \lambda \det A$ for all possible choices of A.

[4] 8. Let A be a 3×3 matrix and let $\det A = -2$.

(a) Find $\det\left(A^T A^2 (-2A)^{-1}\right)$.

(b) Find $\det\left(\operatorname{adj}(2A)\right)$.

[6] 9. (a) Find matrices W, X, Y and Z such that $\begin{bmatrix} O & A \\ B & O \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$ (where A and B are invertible matrices).

(b) Use the above result to find C^{-1}, where $C = \begin{bmatrix} 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

[3] 10. Use Cramer's Rule to solve the system:

$$\begin{aligned} 7x - 9y &= 11 \\ 4x + 5y &= -2 \end{aligned}$$

[3] 11. Simplify the matrix expression $(B(B+I)^{-1})^{-1} - B^{-1}$

[6] 12. Given $A = \begin{bmatrix} 3 & 6 & 2 & 1 & 5 & 2 & 2 \\ 1 & 2 & 1 & 0 & 2 & 0 & -3 \\ 1 & 2 & 0 & 1 & 1 & 1 & 4 \\ 1 & 2 & 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 3 & 0 & -1 \end{bmatrix} \sim R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Row A is a subspace of $\mathbb{R}^n$ for what value of n?

(b) Without calculation, give a basis for Row A.

(c) Col A is a subspace of $\mathbb{R}^m$ for what value of m?

(d) Without calculation, give a basis for Col A.

(e) What is rank A^T?

(f) What is dim Nul A^T?

[4] 13. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0 \right\}$.

(a) Is $\mathbf{0}$ in W? Justify your answer.

(b) Is W closed under scalar multiplication? Justify your answer.

(c) Is W closed under vector addition? Justify your answer.

(d) Is W a subspace of $\mathbb{R}^2$? Explain.

[3] 14. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find a 2×2 matrix B such that $AB = O$ but $BA \neq O$ (where O is the zero matrix).

[6] 15. In question 14 you saw that there can be non-zero $n \times n$ matrices A and B such that $AB = O$ but $BA \neq O$. Now let A and B be any two such matrices.

(a) Show that each column of B is in Nul A.

(b) Show that even though $BA \neq O$, it must be true that $(BA)^2 = O$.

(c) Show that B is not invertible.

[4] 16. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let V be the set of all 2×2 matrices X such that $AX = O$. Given that V is a vector space, find a basis for V.

[4] 17. $V = \{p(x) \in \mathbb{P}_2 : p(2) = 0\}$ is a vector space. Find a basis for V and determine the dimension of V.

[2] 18. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) What is the dimension of the range of T if T is a one-to-one mapping? Explain.

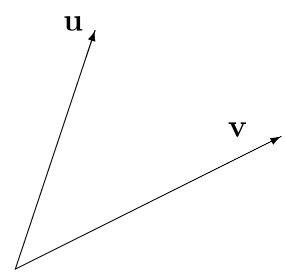
(b) What is the dimension of the kernel of T if T is onto? Explain.

[4] 19. (a) Draw $\{(1-t)\mathbf{u} + t\mathbf{v} \ : \ 0 \leq t \leq 1\}$

(b) Draw $\{s\mathbf{u} + t\mathbf{v} \ : \ 0 \leq s \leq 1, 0 \leq t \leq 1/2\}$

(c) Draw $\text{Proj}_{\mathbf{v}}\mathbf{u} + \text{Proj}_{\mathbf{u}}\mathbf{v}$.

(d) Draw $\text{Proj}_{\mathbf{v}}\mathbf{u} - \text{Perp}_{\mathbf{v}}\mathbf{u}$.



20. Let $\mathcal{L}$ be the line given by $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

[1] (a) Plot $\mathcal{L}$.

[3] (b) Find the distance from $\mathcal{L}$ to the origin.

[2] (c) For what a and b will the the line $\mathbf{x} = \begin{bmatrix} 1 \\ a \end{bmatrix} + t \begin{bmatrix} 1 \\ b \end{bmatrix}$ be the same line as $\mathcal{L}$?

[2] (d) Where does $\mathcal{L}$ intersect the x-axis?

[2] (e) What is the cosine of the angle between $\mathcal{L}$ and the x-axis?

21. Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a plane in $\mathbb{R}^3$.

[3] (a) Find an equation in the form $ax + by + cz = d$ for this plane.

[2] (b) Find an equation for a line through the origin perpendicular to this plane.

[1] (c) For what k is $\begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$ part of this plane?

[3] 22. Give two different unit vectors that are orthogonal to both $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

[3] 23. Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in $\mathbb{R}^3$. Show that if $\dfrac{1}{\mathbf{u} \cdot \mathbf{v}}(\mathbf{u} \times \mathbf{v})$ is a unit vector then the angle between $\mathbf{u}$ and $\mathbf{v}$ is $45°$ or $135°$.

Answers

1. (a) $\mathbf{x} = s\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

   (b) Use part (a) and give non-zero values to s and/or t to generate a set of weights; for instance $s = 1$ and $t = 1$ gives $\mathbf{x} = (1, -1, 1, 1, 0)$:
   $$\begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ -10 \\ -7 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 10 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2. $p(x) = 14 - 8x + \frac{1}{2}x^2$

3. (a) $k \neq -1, 3$ (b) $k = -1, 3$ (c) no value of k

4. (a) $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ (b) $\dfrac{2}{3}$ (c) i. Domain is $\mathbb{R}^3$, codomain is $\mathbb{R}^2$ ii. $\begin{bmatrix} -3 & 4 & 5 \\ -1 & 1 & 1 \end{bmatrix}$

5. You'll want to show that $a_1\mathbf{u} + a_2\mathbf{x} + a_3\mathbf{y} = \mathbf{0}$ has no non-trivial solution. Making the substitutions for $\mathbf{x}$ and $\mathbf{y}$ and rearranging, the equation becomes $(a_1 + 2a_2)\mathbf{u} + (3a_2 + 2a_3)\mathbf{w} + a_3\mathbf{v} = \mathbf{0}$. Since the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, all of the weights in the second equation much be zero. Use standard linear algebra techniques to show that the system of equations
   $\begin{cases} a_1 + 2a_2 \qquad = 0 \\ \qquad 3a_2 + 2a_3 = 0 \\ \qquad\qquad a_3 = 0 \end{cases}$ has only the trivial solution.

6. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

7. (a) $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 5 & 7 & 8 \end{bmatrix}$ (b) $\lambda = 9$ (the determinant of C)

8. (a) $-\dfrac{1}{2}$ (b) 2^8

9. (a) $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} O & B^{-1} \\ A^{-1} & O \end{bmatrix}$ (b) $C^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ -1/2 & 3/2 & 0 & 0 & 0 \end{bmatrix}$

10. $x = \dfrac{37}{71}$, $y = -\dfrac{58}{71}$

11. The expression simplifies to I.

12. a) $n = 7$ b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$

    c) $m = 5$ d) b) $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

    e) 4 f) 1

13. W contains $\mathbf{0}$ and is closed under scalar multiplication, but is not closed under vector addition and therefore is not a subspace of $\mathbb{R}^2$.

14. Multiple answers are possible. One example is $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

15. (a) Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be the columns of B.
    $AB = [A\mathbf{b}_1\ A\mathbf{b}_2] = [\mathbf{0}\ \mathbf{0}] = O$
    Since $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$, $\mathbf{b}_1$ and $\mathbf{b}_2$ are in $\operatorname{Nul} A$.

    (b) Proof:
    $$\begin{aligned} (BA)^2 &= (BA)(BA) & \\ &= B(AB)A & \text{(associativity)} \\ &= BOA & (\text{since} AB = 0) \\ &= O & \end{aligned}$$

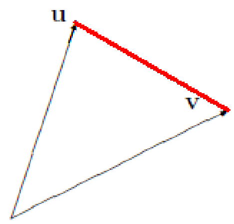
    (c) Suppose B were invertible. Then we could do this:
    $$\begin{aligned} AB &= O \\ ABB^{-1} &= OB^{-1} \\ A &= O \end{aligned}$$
    But $A \neq O$. Contradiction. Therefore B is not invertible.

16. Multiple answers are possible. One example is $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$.

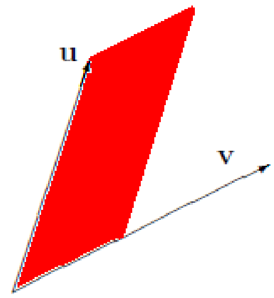
17. $\dim V = 2$. One example of a basis for V is $\{x - 2, x^2 - 4\}$.

18. a) n b) $n - m$

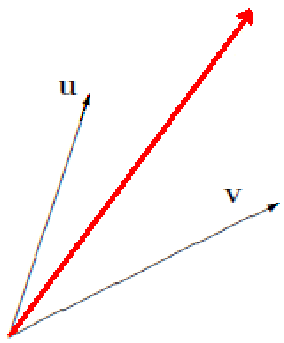
19. a)



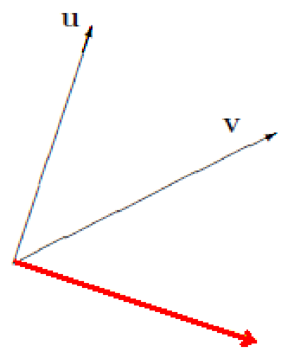
b)



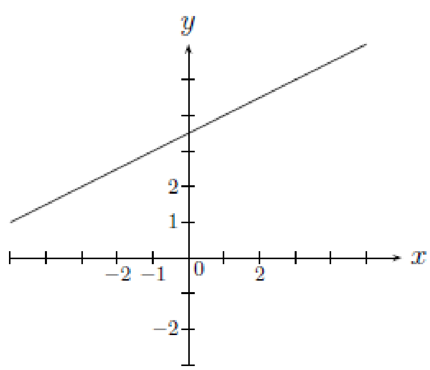
c)



d)



20. a)



b) $\dfrac{7\sqrt{5}}{5}$

c) $a = 4$, $b = 1/2$

d) $(-7, 0)$

e) $\dfrac{2\sqrt{5}}{5}$

21. a) $-x + 2y + 3z = 0$ b) $\mathbf{x} = t\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ c) $k = -1$

22. $\dfrac{1}{5\sqrt{6}}\begin{bmatrix} -1 \\ 10 \\ -7 \end{bmatrix}$ and $\dfrac{-1}{5\sqrt{6}}\begin{bmatrix} -1 \\ 10 \\ -7 \end{bmatrix}$

23. $\left|\left|\dfrac{1}{\mathbf{u}\cdot\mathbf{v}}(\mathbf{u}\times\mathbf{v})\right|\right|$, so $\left|\dfrac{1}{\mathbf{u}\cdot\mathbf{v}}\right| ||(\mathbf{u}\times\mathbf{v})|| = 1$, so $||(\mathbf{u}\times\mathbf{v})|| = |\mathbf{u}\cdot\mathbf{v}|$.

It is always true that $||(\mathbf{u}\times\mathbf{v})|| = ||\mathbf{u}||\,||\mathbf{v}||\sin\theta$, so now we see that $|\mathbf{u}\cdot\mathbf{v}| = ||\mathbf{u}||\,||\mathbf{v}||\sin\theta$, so $\dfrac{|\mathbf{u}\cdot\mathbf{v}|}{||\mathbf{u}||\,||\mathbf{v}||} = \sin\theta$.

Since we also know that $\dfrac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||} = \cos\theta$, we now have $\pm\cos\theta = \sin\theta$, or $\tan\theta = \pm 1$, so finally $\theta = 45^\circ$ or 135°.