(Marks)

1. Given the following homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 0 & 2 & -1 & 0 \\ 1 & 1 & -5 & 5 & 1 \\ 2 & 2 & -10 & 10 & 3 \\ 2 & 1 & -7 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   (a) Write the solution to the system in parametric vector form.

   (b) Write the zero vector in $\mathbb{R}^4$ as a nontrivial linear combination of the columns of A.

   (c) Solve the system $A\mathbf{x} = \mathbf{a}_3$ where $\mathbf{a}_3$ is the third column of the matrix A.

2. Use techniques of linear algebra to find a polynomial $p(x) = a_0 + a_1x + a_2x^2$ such that $p(2) = 0$, $p(-2) = 32$ and $p'(1) = -7$.

3. Let S be a set of vectors. In one short sentence, define what is meant by the *span of* S.

4. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ k \\ 2k+3 \end{bmatrix}$ and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. For what value(s) of k is:

   (a) Span(S) all of $\mathbb{R}^3$?

   (b) Span(S) a plane in $\mathbb{R}^3$?

   (c) Span(S) a line in $\mathbb{R}^3$?

5. Suppose that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, and that $\mathbf{x} = 2\mathbf{u} + 3\mathbf{w}$ and $\mathbf{y} = \mathbf{v} + 2\mathbf{w}$. Prove that the set $\{\mathbf{u}, \mathbf{x}, \mathbf{y}\}$ is linearly independent.

6. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+2y \\ 2x-3y \end{bmatrix}$.

   (a) Find the standard matrix for T_1.

   (b) Is T_1 one-to-one or onto?

   (c) If $\mathbb{L}$ is the line $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ k \end{bmatrix}$, then for what value(s) of k, will $T_1(\mathbb{L})$ be a horizontal line in $\mathbb{R}^2$?

   (d) Now suppose that the composition $T_1 \circ T_2$ is also a linear transformation whose standard matrix is $\begin{bmatrix} 1 & -2 & -3 \\ -3 & 5 & 7 \end{bmatrix}$.

      i. If $T_2 : \mathbb{R}^m \to \mathbb{R}^n$ then what is m? What is n?

      ii. Find the standard matrix for T_2.

7. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ -4 & 1 & -7 \\ 6 & -9 & 1 \end{bmatrix}$.

   (a) Find an LU factorization of A, where L is unit lower triangular and U is upper triangular.

   (b) Find elementary matrices E_1, E_2 and E_3 such that $E_3E_2E_1A = U$.

   (c) Find the determinant of A.

8. Let A be a 4×4 matrix and let $\det A = -2$.

   (a) Find $\det(B)$ where B is a matrix obtained from A by interchanging the second and third rows, then multiplying the first row by 6.

   (b) Find $\det(2R)$ where R is the reduced row echelon form of A.

   (c) Find $\det(A^TA^2(A)^{-1})$.

   (d) Find $\det(\operatorname{adj}(A))$.

9. Use Cramer's Rule to solve the system:

$$\begin{aligned} 7x - 9y &= 11 \\ 4x + 5y &= -2 \end{aligned}$$

(Marks)

10. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 5 \\ 1 & 1 & 8 \end{bmatrix}$.

(a) Find A^{-1}. Verify your answer.

(b) Use A^{-1} to solve the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$.

(c) Find matrices W, X, Y and Z such that $\begin{bmatrix} O & A \\ 3A^T & O \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$.

11. Given $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 & -1 \\ 1 & 2 & 1 & 0 & 2 & 0 & -3 \\ 1 & 2 & 0 & 1 & 1 & 1 & 4 \\ 1 & 2 & 1 & 0 & 2 & 1 & 1 \\ 3 & 6 & 2 & 1 & 5 & 2 & 2 \end{bmatrix} \sim R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Row A.

(b) Find a basis for Col A.

(c) Do either of the vectors $(2, 2, 2, 2, 6)^T$ or $(-5, 1, 1, 2, 1, 0, 0)^T$ belong to Nul A? Justify.

(d) What is rank A^T? What is $\dim(\text{Nul}\, A^T)$?

12. Find a basis and determine the dimension for the vector space $V = \{p(x) \in \mathbb{P}_2 : p(2) = 0\}$.

13. Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy \geq 0 \right\}$.

(a) Is $\mathbf{0}$ in W? Justify your answer.

(b) Is W closed under scalar multiplication? Justify your answer.

(c) Is W closed under addition? Justify your answer.

(d) Is W a subspace of $\mathbb{R}^2$? Explain.

14. Answer each of the following true or false. No justification is required. Given a plane $\mathcal{P}$ in $\mathbb{R}^3$ and a point Q not on the plane $\mathcal{P}$:

(a) There is exactly one plane parallel to $\mathcal{P}$ containing Q.

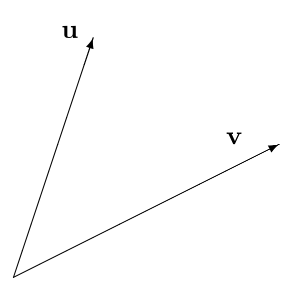
(b) There is exactly one line parallel to $\mathcal{P}$ containing Q.

(c) There is exactly one plane orthogonal to $\mathcal{P}$ containing Q.

(d) There is exactly one line orthogonal to $\mathcal{P}$ containing Q.

15. On the diagrams provided, draw:

(a) $\mathbf{u} + t\mathbf{v}\ :\ t \in \mathbb{R}$



(b) $\mathbf{u} - \text{Proj}_{\mathbf{v}}\mathbf{u}$

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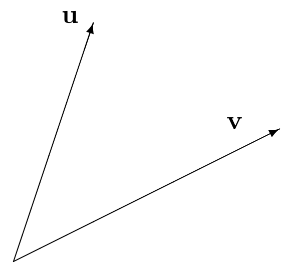

16. Define the line $\mathcal{L}: \begin{bmatrix} -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

    (a) Find the distance from $\mathcal{L}$ to the origin.

    (b) For what a and b will the the line $\mathbf{x} = \begin{bmatrix} 1 \\ a \end{bmatrix} + t \begin{bmatrix} 1 \\ b \end{bmatrix}$ be the same line as $\mathcal{L}$?

17. Recall that $||\mathbf{v}||^2 = \mathbf{v}^T\mathbf{v}$. Suppose that A is the standard matrix for a transformation T and suppose that $A^T = A^{-1}$. Show that the magnitude of every vector is preserved by the transformation T i.e prove that $||T(\mathbf{x})|| = ||\mathbf{x}||$ for all x.

18. If $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ then Span $\{\mathbf{u}, \mathbf{v}\}$ is a plane (denoted $\mathcal{P}$) in $\mathbb{R}^3$.

    (a) Is $\mathcal{P}$ a subspace of $\mathbb{R}^3$? Justify your answer in one short sentence.

    (b) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

    (c) Find an equation for $\mathcal{P}$ in the form $ax + by + cz = d$.

    (d) Find the intersection of $\mathcal{P}$ with the line containing the point $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and parallel to the vector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

ANSWERS: 1. a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ b) $\mathbf{0} = 2\mathbf{a}_1 + 3\mathbf{a}_2 + \mathbf{a}_3 + 0\mathbf{a}_4 + 0\mathbf{a}_5$ (This answer is not unique. It is obtained using $x_3 = 1$ and $x_4 = 0$.) c) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ 2. $p(x) = \frac{x^2}{2} - 8x + 14$ 3. The *Span* of S is the set of all linear combinations of the vectors in S. 4. a) $k \neq 3, k \neq -1$ b) $k = 3, k = -1$ c) Impossible 5. Suppose $c_1\mathbf{u} + c_2\mathbf{x} + c_3\mathbf{y} = \mathbf{0}$. Then $c_1\mathbf{u} + c_2(2\mathbf{u} + 3\mathbf{w}) + c_3(\mathbf{v} + 2\mathbf{w}) = \mathbf{0}$. Then $(c_1 + 2c_2)\mathbf{u} + c_3\mathbf{v} + (3c_2 + 2c_3)\mathbf{w} = \mathbf{0}$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, this means $c_1 + 2c_2 = 0$, $c_3 = 0$, $3c_2 + 2c_3 = 0$ which can only be true if $c_1 = c_2 = c_3 = 0$. Therefore $\{\mathbf{u}, \mathbf{x}, \mathbf{y}\}$ is linearly independent. 6. a) $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ b) T is $1-1$ and onto. c) $k = 2/3$ d)-i. $m = 3$, $n = 2$ d)-ii. $\begin{bmatrix} -3 & 4 & 5 \\ -1 & 1 & 1 \end{bmatrix}$ 7. a) $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & -17 \end{bmatrix}$ b) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ (E_1 and E_2 may be switched) c) det$A$=det$U$ $= -170$ 8. a) 12 b) 16 c) 4 d) -8 9. $x = 37/71$, $y = -58/71$ 10. a) $A^{-1} = \begin{bmatrix} -13 & 9 & -4 \\ -11 & 7 & -3 \\ 3 & -2 & 1 \end{bmatrix}$ b) $\mathbf{x} = \begin{bmatrix} 36 \\ 27 \\ -8 \end{bmatrix}$ c) $W = 0$, $X = \frac{1}{3}(A^{-1})^T$, $Y = A^{-1}$,

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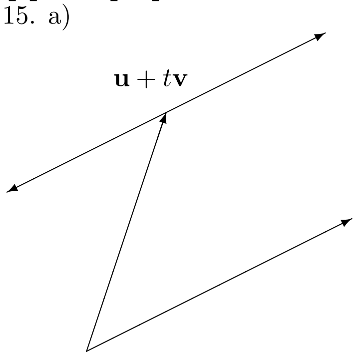
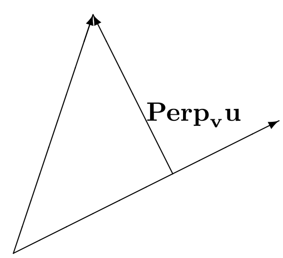
$Z = 0$ 11. a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$ b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ c) Yes, $(-5, 1, 1, 2, 1, 0, 0)^T \in \text{Nul}A$ d) $\text{rank}A^T = 4$, $\dim(\text{Nul}A^T) = 1$ 12. $\mathcal{B} = \{x^2 - 4, x - 2\}$, $\dim V = 2$ 13. a) Yes b) Yes. If $\begin{bmatrix} x \\ y \end{bmatrix} \in W$ then $\begin{bmatrix} kx \\ ky \end{bmatrix} \in W$ since $(kx)(ky) = k^2(xy)$ which is greater than or equal to 0 because $k^2 \geq 0$ and $xy \geq 0$. c) No, since $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$ are both in W but their sum is not. d) No 14. a) True b) False c) False d) True

15. a)



b)



16.a) $\frac{\sqrt{245}}{5}$ b) $a = 4$, $b = \frac{1}{2}$ 17. $||T(\mathbf{x})||^2 = ||A\mathbf{x}||^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T A^{-1} A\mathbf{x} = \mathbf{x}^T\mathbf{x} = ||\mathbf{x}||^2$. Since $||\mathbf{x}|| \geq 0$, if $||T(\mathbf{x})||^2 = ||\mathbf{x}||^2$ then $||T(\mathbf{x})|| = ||\mathbf{x}||$. 18. a) Yes, it is a span of some set of vectors. b) $\frac{1}{\sqrt{15}}$ c) $-x + 2y + 3z = 0$ d) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$