(6) 1. Given
$$\begin{array}{rcl} -x_1 + x_2 + 5x_3 + 8x_4 - 6x_5 &=& 1 \\ -x_2 - 2x_3 - 3x_4 + 4x_5 &=& -2 \\ 2x_1 \qquad - 6x_3 - 10x_4 + 4x_5 &=& k \end{array}$$

(a) Find the value of k which makes this system consistent.

(b) Using the value of k from part **a**, find the general solution to the system of equations and express it in parametric vector form.

(4) 2. Given $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 13 \\ 16 \end{bmatrix}$, determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. If not, give a non-trivial dependence relation between $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

(4) 3. Use the matrix method to balance the chemical equation $C_4H_{10} + O_2 \rightarrow CO_2 + H_2O$.

(5) 4. Let $A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & k & 4 \\ 3 & -2 & 0 & -2 \\ -2 & 2 & 3 & 4 \end{bmatrix}$.

(a) Find the determinant of A.

(b) What are the possible value(s) of k such that $\det(A) = \det(A^{-1})$?

(3) 5. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ such that $\det A = 10$.

(a) Find the determinant of $\begin{bmatrix} 2a & 2b & 2c \\ 5g & 5h & 5i \\ d & e & f \end{bmatrix}$.

(b) Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be the columns of A. Find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

(4) 6. Show that the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - |y| \end{bmatrix}$ is not a linear transformation.

(10) 7. Let S be the transformation given by $S(\mathbf{x}) = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}\mathbf{x}$, and let T be the transformation given by $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}\mathbf{x}$.

(a) Is S a one-to-one transformation? Justify your answer.

(b) Find a vector $\mathbf{x}$ such that $S(\mathbf{x}) = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(c) What is the range of T? Justify your answer.

(d) Find a non-zero vector inside the kernel of T.

(e) Let $R(\mathbf{x}) = S(T(\mathbf{x}))$. Find the standard matrix of the transformation R.

(4) 8. Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which rotates any vector $\frac{3\pi}{4}$ radians counterclockwise about the origin and then reflects it across the line $y = x$. Find the standard matrix of T.

(4) 9. Let $ABC = I$ where A, B, and C are all square.

(a) Prove that A, B, and C are invertible.

(b) Find B^{-1}.

(2) 10. Let $X = D^T D + I$. Prove that $X^T = X$.

(4) 11. Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Find A^{-1}.

(3) 12. The 2×2 matrix A can be row-reduced to I by the following elementary row operations (in order):

- Swap row 1 and row 2.
- Multiply row 2 by $\frac{1}{2}$
- Replace row 1 with the sum of itself and -4 times row 2.
- Multiply row 1 by $\frac{1}{3}$

(a) Express A as a product of elementary matrices.

(b) What is $\det A$?

(4) 13. Find an LU factorization of $A = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -4 & 5 & 3 \\ -6 & 13 & 4 & -6 \end{bmatrix}$.

(4) 14. Given the following block matrix equation: $\begin{bmatrix} O & A \\ I & B \end{bmatrix} \begin{bmatrix} X & Y \\ Z & O \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$.

Assume A is invertible.

(a) Find X, Y, and Z in terms of A, B, and I.

(b) Use your answer in part (a) to find the inverse of $M = \begin{bmatrix} 0 & 0 & 5 & 8 \\ 0 & 0 & 2 & 3 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 6 \end{bmatrix}$.

(6) 15. Let $\mathbf{b}$ be a fixed vector in $\mathbb{R}^n$ and let H be the set of all $m \times n$ matrices A such that $A\mathbf{b} = \mathbf{0}$.

That is, $H = \{A : A \text{ is an } m \times n \text{ matrix and } A\mathbf{b} = \mathbf{0}\}$

(a) Prove that H is a subspace of $M_{m\times n}$.

(b) In particular, let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find a basis for the set of all 2×2 matrices such that $A\mathbf{b} = \mathbf{0}$.

That is, find a basis for $H = \left\{ A : A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

(4) 16. Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : xy = z^2 \right\}$.

Prove that S has the stated property or use a counterexample to show that the property fails.

(a) S is closed under vector addition.

(b) S is closed under multiplication by scalars.

(8) 17. Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & -3 \\ 1 & 2 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find a basis for $\operatorname{Col} A$.

(b) Find a basis for $\operatorname{Nul} A$.

(c) Find a basis for $\operatorname{Row} B$.

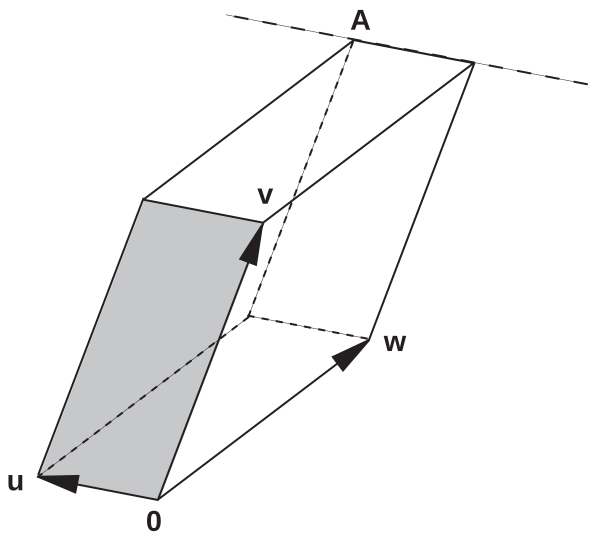
(d) Prove that $\operatorname{Col} A = \operatorname{Row} B$.

(4) 18. Suppose that A is an $n \times n$ matrix such that $\operatorname{Nul} A = \operatorname{Col} A$.

(a) Prove that n must be even.

(b) Prove that $A^2 = 0$.

(7) 19. The following picture shows the parallelepiped formed by the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$.



(a) Find the area of the shaded face.

(b) Find an equation of the plane containing the shaded face.

(c) Find the volume of the parallelepiped.

(d) Find the coordinates of the point labeled A in the diagram.

(e) Find an equation for the line through the upper back edge shown in the picture.

(5) 20. Let $\mathcal{L}_1$ be the line $\mathbf{x} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and let $\mathcal{L}_2$ be the line $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find the distance between the skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Find the point on $\mathcal{L}_2$ that is closest to the origin.

(5) 21. Let $\mathbf{u} = \begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ 1 \end{bmatrix}$.

(a) Find a unit vector orthogonal to $\mathbf{u}$.

(b) Find $\text{Proj}_{\mathbf{v}}\mathbf{u}$.

(c) Find $\|\text{Proj}_{\mathbf{v}}\mathbf{u}\|$.