Answers to Math 803-Final Exam (December 2011)

1. A permutation without repetition: ${}_7P_4 = \frac{7!}{3!} = 840$

2. A combination with repetition: ${}_9C_7 = \frac{9!}{7!2!} = 36$

3. (a) 2^{10}
   (b) ${}_{10}C_8 = 45$
   (c) A has 1 subset with 10 elements and 10 subsets each with 9 elements and 45 subsets with 8 elements. Thus A has $2^{10} - 45 - 10 - 1 = 968$ elements with fewer than 8 elements.

4. (a) $\{1, 2, 3, 4\}$
   (b) $\{1, 4\}$
   (c) $\{1, 2, 4\}$
   (d) $\{2, 3, 5, 6\}$

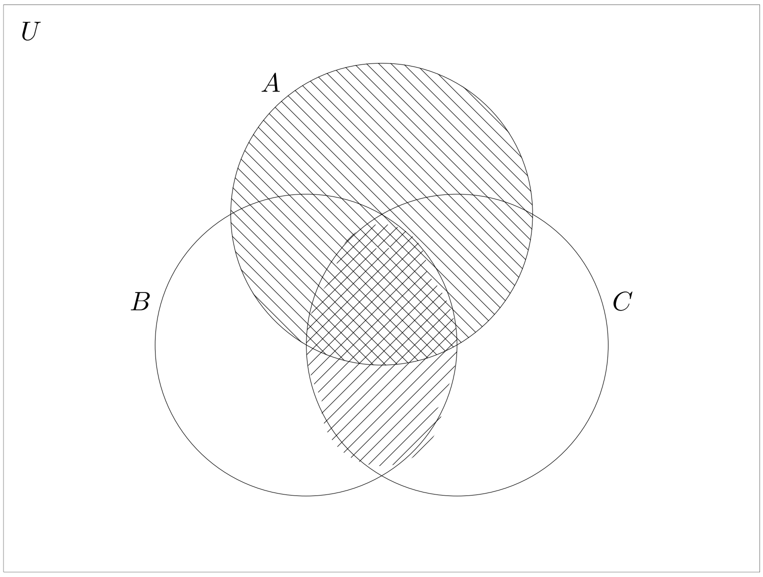
5. $\{x \in \mathbb{N} : \ x = 2n \text{ for some } n \in \mathbb{N}\}$

6. (a) If I am late for school then I miss the bus.
   (b) If I don't miss the bus then I am not late for school.
   (c) If I am not late for school then I don't miss the bus.
   (d) Yes; statement (c), the contrapositive, is logically equivalent to the original.

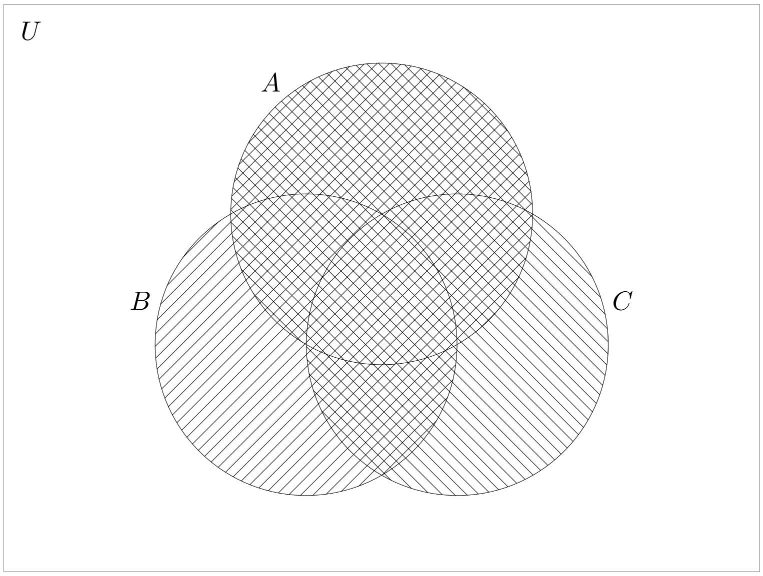
7. The statement is false in the four cases where p is true and the statement is true in the four cases where p is false. Thus, the statement is neither a tautology nor a contradiction.

8. Both statements are false when p is false and q is true. Both statements are true in all other circumstances. Thus the two statements are logically equivalent.

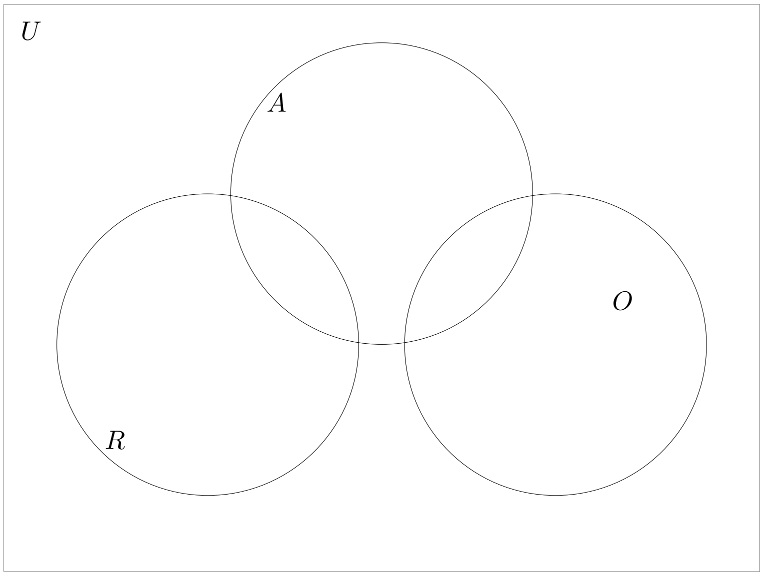
9. A Venn diagram for $A \cup (B \cap C)$: The entire hatched area is $A \cup (B \cap C)$.



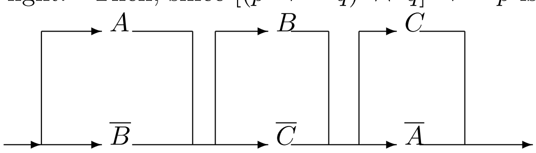
A Venn diagram for $(A \cup B) \cap (A \cup C)$ is: The cross-hatched area is $(A \cup B) \cap (A \cup C)$.



10. The argument is invalid. Let R represents red things, let A represent apples and let O represent oranges. Then the following is a Venn diagram satisfying both hypotheses, but not the conclusion, of the argument.



11. Let p be the statement "General relativity is correct" and let q be the statement "Neutrinos can travel faster than light." Then, since $[(p \to \sim q) \ \wedge \ q] \to \sim p$ is a tautology, the argument is valid.

12. (a) 

    (b) 0

13. (a)

$$\begin{aligned} A(B+\overline{B})+\overline{A}+B &= A\cdot 1+\overline{A}+B \\ &= A+\overline{A}+B \\ &= 1+B \\ &= 1 \end{aligned}$$

(b)

$$
\begin{aligned}
(\overline{A+B})\overline{A} + A\overline{B} + C\overline{C} &= (\overline{A+B})\overline{A} + A\overline{B} + 0 \\
&= (\overline{A+B})\overline{A} + A\overline{B} \\
&= (\overline{A}\overline{B})\overline{A} + A\overline{B} \\
&= (\overline{A}\overline{A})\overline{B} + A\overline{B} \\
&= \overline{A}\overline{B} + A\overline{B} \\
&= (\overline{A} + A)\overline{B} \\
&= 1 \cdot \overline{B} \\
&= \overline{B}
\end{aligned}
$$

14.

A	B	$A+B$	AB	$\overline{AB}$	$(\overline{AB})(A+B)$
1	1	1	1	0	1
1	0	1	0	1	1
0	1	1	0	1	1
0	0	0	0	1	0

15. One such expression is: $AB\overline{C} + A\overline{B}C + \overline{A}BC$

16. $\overline{AB} = \overline{A} + \overline{B}$ or $\overline{A+B} = \overline{A}\ \overline{B}$

17. $A \cup (B \cup C) = (A \cup B) \cup C$

18. Matrix multiplication is not commutative. For example, if A is 3×2 and B is 2×3 then the matrices AB and BA are not even the same size.

19. a) Dependent; Consistent with infinitely many solutions b) Independent; Inconsistent.

20. The solution to the system is $x = -2$, $y = 1$.

21. a) $C^T - 3I = \begin{bmatrix} -2 & 7 & -2 \\ -3 & -2 & 4 \\ 2 & 0 & 2 \end{bmatrix}$ b) $A^{-1} = \frac{1}{16}\begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix}$ c) B^{-1} does not exist

d) $A^2 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$ e) Impossible f) Impossible g) $EA = \begin{bmatrix} -16 & 0 \\ -7 & 5 \\ 2 & -6 \end{bmatrix}$

22. $A^{-1} == \begin{bmatrix} 0 & 3 & 1 \\ 3 & -8 & 2 \\ 2 & -6 & 1 \end{bmatrix}$

23. a) $\begin{bmatrix} -4 & -3 & 2 \\ 4 & 3 & -1 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ b) $x = -6$, $y = 8$, $z = 1$

24. a) $x = 1 + 3y$, $y =$ free, $z = 5$ b) No solution

25. $x = 15$, $y = 1$, $z = -3$

26. $x = 5 - z$, $y = -3 + 2z$, $z =$ free

27. Base Step: $1^3 = \dfrac{1^2(1+1)^2}{4}$

Induction Step: Assume that $1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$. Then

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$