1. Solve the system or show that it is inconsistent.

[4] (a) $\begin{cases} x_1 + 2x_2 + 3x_4 = 16 \\ -2x_1 - 4x_2 + x_3 - 8x_4 = -39 \\ x_1 + 2x_2 + 2x_3 = 7 \end{cases}$

[4] (b) $\begin{cases} 2x_1 - 6x_2 = 8 \\ -3x_1 + 4x_2 = 3 \\ 2x_1 + x_2 = -13 \end{cases}$

[5] 2. Given the system: $\begin{cases} x - 2y + 3z = s \\ -4y + (r+6)z = 2s \\ 4y - 8z = 6-s \end{cases}$

Find a condition on r and s such that the system has

(a) infinitely many solutions.

(b) no solutions.

(c) a unique solution.

[5] 3. The weekly production of a furniture manufacturer is described by the following system of equations:
$\begin{cases} 6x + 2y + 7z = 60 \\ 12x + 6y + 2z = 48 \end{cases}$
where x, y, and z are respectively the number of chairs, tables, and sofas produced.

(a) Find the general solution to the system.

(b) Find ALL particular solutions that are realistic.

[7] 4. Given:

$A = \begin{bmatrix} 3 & 8 \\ -1 & 4 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 7 & 1 \\ 0 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 3 & 8 \\ -5 & 0 & 1 \end{bmatrix}$, and $E^{-1} = \begin{bmatrix} 4 & 0 \\ 1 & -1 \end{bmatrix}$;

Find the following or state that they are undefined.

(a) C^2

(b) $3A^T - DC$

(c) B

(d) $(EB)^{-1}$

[9] 5. Let $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & -2 \\ 6 & 12 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.

(a) Express the matrix equation $A\mathbf{x} = \mathbf{b}$ as a system of linear equations in the variables x, y, and z.

(b) Find $\text{adj}(A)$.

(c) Multiply $A \cdot \text{adj}(A)$

(d) Find $\det(A)$.

(e) Find A^{-1}.

(f) Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ using A^{-1}.

[2] 6. Let A and B be 3×3 matrices such that $\det(A) = -4$ and $\det(B) = 7$. Find the following:

(a) $\det(-2B^T)$

(b) $\det[(BA)^{-1}]$

[2] 7. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a matrix such that $\det(A) = -3$. Find the determinant of each matrix below:

(a) $B = \begin{bmatrix} 2d+5a & 2e+5b & 2f+5c \\ d & e & f \\ g & h & i \end{bmatrix}$

(b) $C = \begin{bmatrix} a-d & b-e & c-f \\ g & h & i \\ d-a & e-b & f-c \end{bmatrix}$

[6] 8. Given the system:

$$\begin{cases} 2x + y - 2z - 2w = 0 \\ 4x - y + 3z + 5w = 0 \\ 3x + 5y - 3z - 4w = 7 \\ -x - 2z + 2w = 0 \end{cases}$$

Solve for w only using Cramer's Rule.

[6] 9. Let a simple economy have two industries: Pulp and Juice. To produce \$1 of pulp requires 70¢ of pulp and 40¢ of juice. To produce \$1 of juice requires 10¢ of pulp and 20¢ of juice.

(a) If there is an external demand for \$2500 of pulp and \$3000 of juice, how much of each should be produced to meet it?

(b) Find the internal consumption of the economy.

(c) Which, if any, of the two industries is profitable? Justify.

[5] 10. Given points $A(3,1,0,-2)$, $B(7,1,2,-5)$, $C(7,0,-1,0)$, and $D(11,1,1,-3)$;

(a) Find the vector $\overrightarrow{AB}$.

(b) Find the magnitude $|\overrightarrow{AB}|$.

(c) Find *parametric equations* for the line L that passes through C and is parallel to $\overrightarrow{AB}$.

(d) Is the point D on the line L? Justify your answer.

[4] 11. Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{u_3} = \begin{bmatrix} 6 \\ 8 \\ -7 \end{bmatrix}$.

Express the vector $\mathbf{u_3}$ as a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$ if possible.

[2] 12. Suppose A is a 3×3 matrix with columns $\mathbf{a_1}$,$\mathbf{a_2}$, and $\mathbf{a_3}$, and let $\mathbf{b}$ be another vector in $\mathbb{R}^3$.

(a) If $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x_0} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$, express $\mathbf{b}$ as a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$, and $\mathbf{a_3}$.

(b) Given that $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, find another solution to $A\mathbf{x} = \mathbf{b}$.

13. Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{a_4}\ \mathbf{a_5}]$ be a 4×5 matrix with columns $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, $\mathbf{a_4}$, and $\mathbf{a_5}$.

Suppose A has reduced form $R = \begin{bmatrix} 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

[2] (a) If $\mathbf{a_1} = \begin{bmatrix} 7 \\ 3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 5 \end{bmatrix}$, and $\mathbf{a_5} = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 4 \end{bmatrix}$, find $\mathbf{a_3}$ and $\mathbf{a_4}$.

[3] (b) For each of the following sets, determine if it is linearly dependent (LD) or linearly independent (LI). In each case, state the dimension of the span.

i. $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$

ii. $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_4}\}$

iii. $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_4}, \mathbf{a_5}\}$

[2] (c) Find a basis for $\text{Nul}(A)$.

[2] 14. Let A be a 5×5 matrix with $\det(A) = 7$.
Determine whether the following are True or False.

(a) The column space of A has dimension 5.

(b) If $Ax = Ay$, then $x = y$.

[2] 15. Let A be a 7×10 matrix.

(a) Is it possible that $A\mathbf{x} = \mathbf{0}$ has a unique solution? Explain.

(b) If the rank of A is 6, how many parameters are there in a solution to $A\mathbf{x} = \mathbf{0}$? Explain.

[6] 16. For each of the sets below,
(i) List two non-zero vectors in the set.
(ii) Determine if the set is a subspace of $\mathbb{R}^3$.
(iii) If the set is a subspace, express it as a span of vectors. If the set is not a subspace, find a counter-example to show it not closed under vector addition or scalar multiplication.

(a) $S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} 3x + 5z = 0 \\ \text{and} \\ y - 2z = 0 \end{matrix} \right\}$.

(b) $S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; z^2 - xy = 0 \right\}$.

[6] 17. Solve the linear program by the simplex method:

$$\begin{array}{llcrcrcrcl} \text{Maximize} & z & = & 5x_1 & + & 7x_2 & + & x_3 & & \\ \text{subject to} & & & x_1 & + & 3x_2 & - & x_3 & \leq & 1 \\ & & & x_1 & + & x_2 & + & x_3 & \leq & 5 \\ & & & x_1 & \geq 0, & x_2 & \geq 0, & x_3 & \geq 0 & \end{array}$$

Give the full basic feasible solution, including the slack variables.

[8] 18. A metal refinery processes two kinds of ores according to the following table.

	Heating	Separating	Purifying	Revenue
Ore I	1	3	1	60
Ore II	2	3	5	90
Hours Available	120	180	240	

For example, processing one ton of Ore I requires 1 hour of heating, 3 hours of separating, and 1 hour of purifying, and brings \$60 in revenue. Similarly for Ore II. The refinery would like determine how much of each ore should be processed in order to maximize revenue.

(a) Name variables and set-up a linear program that represents this situation.

Below you will find the initial and final simplex tables for this program.

$$\left[\begin{array}{c|cc|ccc|c} 1 & -60 & -90 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 1 & 0 & 0 & 120 \\ 0 & 3 & 3 & 0 & 1 & 0 & 180 \\ 0 & 1 & 5 & 0 & 0 & 1 & 240 \end{array}\right] \to \left[\begin{array}{c|cc|ccc|c} 2 & 0 & 0 & 0 & 35 & 15 & 9900 \\ \hline 0 & 0 & 0 & 20 & -5 & -5 & 300 \\ 0 & 12 & 0 & 0 & 5 & -3 & 180 \\ 0 & 0 & 12 & 0 & -1 & 3 & 540 \end{array}\right]$$

Use this information to answer the remaining questions.

(b) What is the maximum revenue?

(c) How much of each ore should be processed in order to maximize revenue?

(d) When revenue is maximized, how many hours each of heating, separating, and purifying will go unused?

19. Consider the following linear program:

$$\begin{array}{lrcrcl} \text{Maximize} & z = & x & + & y & \\ \text{subject to} & & -x & + & 2y & \le 4 \\ & & x & + & 2y & \le 8 \\ & & 2x & - & y & \le 6 \\ & & x & \ge 0, & y & \ge 0 \end{array}$$

[6] (a) Solve the program by sketching the feasibility region. Give both the maximum value of z and the values of x and y where the maximum in attained.

[2] (b) Create the initial simplex table and circle the pivot in this table that leads to the solution along the shortest path. (Do not perform the complete algorithm.)

Answers:

1(a) $(1-2t, t, 3, 5)$ 1(b) $(-5,-3)$ 2(a) $r = 2$ and $s = -6$ 2(b) $r = 2$ and $s \neq -6$

2(c) $r \neq 2$ 3(a) $(22 - 19t/6, -36 + 6t, t)$ 3(b) 3 chairs, 0 tables, and 6 sofas.

4(a) Undefined 4(b) $\begin{bmatrix} -8 & 40 \\ 14 & 34 \end{bmatrix}$ 4(c) $\frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ 4(d) $(EB)^{-1} = B^{-1}E^{-1} = \begin{bmatrix} 11 & -3 \\ 8 & -4 \end{bmatrix}$

5(a) $\left\{\begin{array}{rcrcrcr} 2x & + & 4y & & & = & 2 \\ x & + & 3y & - & 2z & = & -1 \\ 6x & + & 12y & + & z & = & 5 \end{array}\right.$ 5(b) $\operatorname{adj}(A) = \begin{bmatrix} 27 & -4 & -8 \\ -13 & 2 & 4 \\ -6 & 0 & 2 \end{bmatrix}$

5(c) $A \cdot \operatorname{adj}(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 5(d) $\det(A) = 2$ 5(e) $A^{-1} = \frac{1}{2}\begin{bmatrix} 27 & -4 & -8 \\ -13 & 2 & 4 \\ -6 & 0 & 2 \end{bmatrix}$

5(f) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 9 \\ -4 \\ -1 \end{bmatrix}$ 6(a) -56 6(b) $-\dfrac{1}{28}$ 7(a) -15 7(b) 0

8. $w = \dfrac{\det(A_w)}{\det(A)} = \dfrac{-77}{-227} = \dfrac{77}{227}$ 9(a) Total Production: \$11,500 in Pulp and \$9,500 in Juice.

9(b) Internal Consumption: \$9,000 in Pulp and \$6,500 in Juice.

9(c) Juice is profitable as it costs 30¢ to produce \$1. Pulp is not profitable as it costs \$1.10 to produce \$1.

10(a) $\overrightarrow{AB} = (4, 0, 2, -3)$ 10(b) $|\overrightarrow{AB}| = \sqrt{29}$ 10(c) $\begin{cases} x_1 &= 7 + 4t \\ x_2 &= 0 \\ x_3 &= -1 + 2t \\ x_4 &= -3t \end{cases}$ 10(d) No. $x_2 \neq 0$

11. $\mathbf{u_3} = \frac{5}{2}\mathbf{u_1} + \frac{1}{2}\mathbf{u_2}$ 12(a) $\mathbf{b} = 5\mathbf{a_1} - \mathbf{a_2} + 3\mathbf{a_3}$

12(b) General solution is $\mathbf{x} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. So another solution is $\begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$ (when $t = 1$).

13(a) $\mathbf{a_3} = 5\mathbf{a_1} - 2\mathbf{a_2} = \begin{bmatrix} 31 \\ 17 \\ -12 \\ -5 \end{bmatrix}$ and $\mathbf{a_4} = \mathbf{a_5} + \mathbf{a_1} - 2\mathbf{a_2} = \begin{bmatrix} 9 \\ 5 \\ -3 \\ -5 \end{bmatrix}$

13(b)i. L.D. Dimension=2. 13(b)ii. L.I. Dimension=3 13(b)iii. L.D. Dimension=3

13(c) A basis for $\text{Nul}(A)$ is $\left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. 14(a) T 14(b) T

15(a) No, a solution will contain at least three parameters.

15(b) Since $\text{Nul}(A) = 10 - 6 = 4$, there will be exactly 4 parameters in the solution.

16(a)i. $\left\{ \begin{bmatrix} -5 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} -10 \\ 12 \\ 6 \end{bmatrix} \right\}$ 16(a)ii. S_1 is a subspace of $\mathbb{R}^3$. 16(a)iii. $S_1 = \text{Span}\left\{ \begin{bmatrix} -5 \\ 6 \\ 3 \end{bmatrix} \right\}$.

16(b)i. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ 16(b)ii. Not a subspace. 16(b)iii. Since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin S_2$, the set is not closed under vector addition. 17. Max $z = 17$ at $(3, 0, 2, 0, 0)$.

18(a) Let $x_1 =\#$ tons of Ore 1 produced.
Let $x_2 =\#$ tons of Ore 2 produced.

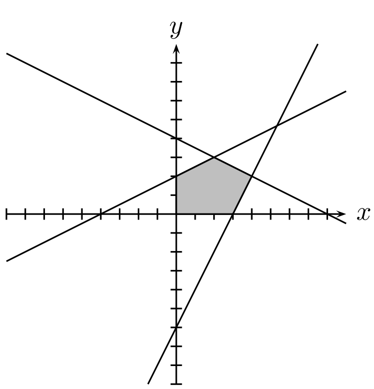
$$\begin{array}{lrcrcl} \text{Maximize} & z = 60x_1 & + & 90x_2 & & \\ \text{subject to} & x_1 & + & 2x_2 & \leq 120 \\ & 3x_1 & + & 3x_2 & \leq 180 \\ & x_1 & + & 5x_2 & \leq 240 \\ & x_1 & \geq 0, & x_2 & \geq 0 \end{array}$$

18(b) Max revenue is \$4950

18(c) 15 tons of Ore I and 45 tons of Ore II should be produced.

18(d) 15 hours of heating will go unused. No hours of heating or separating will remain.

19(a) Max $z = 6$ at $(4, 2)$.



19(b)

$$\left[\begin{array}{c|cc|ccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 & 0 & 8 \\ 0 & \textcircled{2} & -1 & 0 & 0 & 1 & 6 \end{array}\right]$$